JOURNAL OF FUNCTIONAL ANALYSIS 88, 299–341 (1990)

# Wiener Sausage and Self-Intersection Local Times

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*Communicated by Paul Malliavin*

Received November 1987

Let $B=(B_t, t\geqslant 0)$ be a standard Brownian motion in $\mathbb{R}^2$. For every $\varepsilon>0$ and every compact subset $K$ of $\mathbb{R}^2$, the Wiener sausage of radius $\varepsilon$ associated with $K$ is defined as the union of the sets $B_s+\varepsilon K$, $s\in[0,1]$. The present paper gives full asymptotic expansions for the area of the Wiener sausage, when the radius $\varepsilon$ goes to 0. The $k$th term of the expansion is of order $|\log\varepsilon|^{-k}$ and involves a random variable which measures the number of $k$-multiple self-intersections of the process. Such random variables are called (renormalized) self-intersection local times and have been recently introduced and studied by E. B. Dynkin. A self-contained construction of these local times is given, together with a number of new approximations. © 1990 Academic Press, Inc.

## 1. INTRODUCTION

Let $B=(B_t, t\geqslant 0)$ be standard Brownian motion in $\mathbb{R}^2$. For every integer $k\geqslant 1$, the process $B$ has points of multiplicity $k$. Recently several authors, including Dynkin [6–9], Rosen and Yor [20], and Rosen [19], have proposed different methods of defining functionals associated with $k$-multiple points, sometimes called renormalized self-intersection local times. The purpose of the present paper is both to give an elementary self-contained construction of some of Dynkin's functionals and to apply this construction to the asymptotics of the planar Wiener sausage.

Let us first describe our approach on a simple special case. We consider an exponential time $\zeta$ with parameter $\lambda>0$, independent of $B$. For every $y\in\mathbb{R}^2$ and $\varepsilon>0$, let $l_\varepsilon^y$ denote the local time of $B$ on the circle of center $y$ and radius $\varepsilon$, at time $\zeta$. That is, $l_\varepsilon^y$ is the value at time $\zeta$ of the additive functional of $B$ which corresponds to the uniform probability measure on the circle of center $y$ and radius $\varepsilon$. For every $k\geqslant 1$ and $\varepsilon>0$, set

$$\tau_\varepsilon^k\varphi=\frac{1}{k!}\int dy\,\varphi(y)(l_\varepsilon^y)^k,$$

where $\varphi$ is any bounded measurable function from $\mathbb{R}^2$ to $\mathbb{R}$. As $\varepsilon$ goes to



0022-1236/90 $3.00
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0, the field $\tau_\varepsilon^1$ converges towards the occupation field of $B$. For every $k \geqslant 2$, the field $\tau_\varepsilon^k$ blows up as $\varepsilon$ goes to 0, but the following renormalization procedure can be used to give a limiting random field. Let $h_\varepsilon$ denote the conditional expected value of $-l_\varepsilon^y$, knowing that the circle of center $y$ and radius $\varepsilon$ is hit by $B$ before time $\zeta$. Clearly $h_\varepsilon$ does not depend on $y$, and an easy calculation shows that

$$h_\varepsilon = -\frac{1}{\pi} \log \frac{1}{\varepsilon} - \frac{1}{\pi} \left( \frac{\log 2 - \log \lambda}{2} - \gamma \right) + O(\varepsilon),$$

where $\gamma$ denotes Euler's constant. For every $k \geqslant 1$ we consider the polynomial

$$P_\varepsilon^k(x) = \sum_{j=1}^{k} \binom{k-1}{j-1} h_\varepsilon^{k-j} \frac{x^j}{j!}$$

whose leading term is $x^k/k!$, and we set

$$T_\varepsilon^k \varphi = \int dy\, \varphi(y)\, P_\varepsilon^k(l_\varepsilon^y).$$

This definition is motivated by the identity

$$P_\varepsilon^k(l_\varepsilon^y) = \int_{\{0 \leqslant s_1 \leqslant \cdots \leqslant s_k < \zeta\}} l_\varepsilon^y(ds_1) \prod_{j=2}^{k} (l_\varepsilon^y(ds_j) + h_\varepsilon\, \delta_{(s_{j-1})}(ds_j)),$$

where $\delta_{(s)}$ denotes the Dirac measure at $s$. Then

$$\lim_{\varepsilon \to 0} T_\varepsilon^k \varphi = T^k \varphi \tag{1.a}$$

exists in $L^p$-norm for every $p < \infty$. Moreover, the fields $T^k$ coincide with those introduced by Dynkin [7]. The limiting result (1.a) can be extended by replacing local times on circles by more general additive functionals. We thus obtain a wide class of approximations of the fields $T^k$.

Let us now pass to our applications to Wiener sausages. Let $K$ be a compact subset of $\mathbb{R}^2$ with positive logarithmic capacity. For every $\varepsilon > 0$, the Wiener sausage of radius $\varepsilon$ associated with $K$ on the time interval $[0; t]$ is defined by the formula

$$S_\varepsilon^K(0, t) = \bigcup_{0 \leqslant s \leqslant t} (B_s + \varepsilon K),$$

where $B_s + \varepsilon K$ denotes the set $\{B_s + \varepsilon y;\ y \in K\}$. In particular, when $K$ is the closed unit disk $D$, $S_\varepsilon^K(0, t)$ is the tubular neighbourhood of radius $\varepsilon$ of the Brownian path. Now consider the random field

$$S_\varepsilon^K \varphi = \int dy\, \varphi(y)\, I(y \in S_\varepsilon^K(0, \zeta)).$$

A basic result of the present work (Theorem 4.1) gives an asymptotic expansion of $S_\varepsilon^K\varphi$ in terms of the random variables $T^k\varphi$. For any $\varepsilon>0$, set

$$h_\varepsilon^K=-\operatorname{cap}(\varepsilon K)^{-1},$$

where the notation cap$(\cdot)$ refers to the capacity relative to the process $B$ killed at time $\zeta$. It is easily seen that

$$h_\varepsilon^K=-\frac{1}{\pi}\log\frac{1}{\varepsilon}-\frac{1}{\pi}\left(\frac{\log 2-\log\lambda}{2}-\gamma+R(K)\right)+O\left(\varepsilon^2\log\frac{1}{\varepsilon}\right),$$

where $R(K)$ is the logarithm of the logarithmic capacity of $K$ (defined as in [10, p. 252]).

Then, for any bounded measurable function $\varphi$, and any $n\geqslant 1$,

$$S_\varepsilon^K\varphi=-\sum_{k=1}^{n}(h_\varepsilon^K)^{-k}T^k\varphi+R_n(\varepsilon,\varphi), \tag{1.b}$$

where the remainder $R_n(\varepsilon,\varphi)$ satisfies

$$\lim_{\varepsilon\to 0}(h_\varepsilon^K)^{2n}E[(R_n(\varepsilon,\varphi))^2]=0.$$

Taking $\varphi=1$, we get an asymptotic expansion for the area of the sausage $S_\varepsilon^K(0,\zeta)$. It is natural to ask whether such an expansion holds for the area of the sausage $S_\varepsilon^K(0,t)$, for a constant time $t>0$. The answer is yes. The statement of the result involves random variables $T^k(t)$ such that, for every $k\geqslant 1$, $T^k(t)$ coincides with $T^k1$ conditionally on $\{\zeta=t\}$. Then, if $m$ denotes Lebesgue measure in $\mathbb{R}^2$,

$$m(S_\varepsilon^K(0,t))=-\sum_{k=1}^{n}(h_\varepsilon^K)^{-k}T^k(t)+R_n(\varepsilon,t), \tag{1.c}$$

where

$$\lim_{\varepsilon\to 0}(h_\varepsilon^K)^nR_n(\varepsilon,t)=0$$

in $L^2$-norm, and almost surely if $K$ is star-shaped, i.e., $\varepsilon K\subset K$ for $\varepsilon\in[0;1]$. Notice that both $h_\varepsilon^K$ and the random variables $T^k(t)$ depend on the choice of the parameter $\lambda$. However, the different expansions we can deduce from (1.c) by changing the value of $\lambda$ are trivially equivalent. In contrast with (1.b), there is no canonical choice of the parameter $\lambda$ when we consider the sausage on a deterministic time interval.

When $n=1$, (1.c) reduces to the well-known result

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)m(S_\varepsilon^K(0,t))=\pi t, \tag{1.d}$$

in $L^2$-norm and a.s. if $K$ is star-shaped. For $n=2$, we obtain

$$\lim_{\varepsilon \to 0}\left(\log \frac{1}{\varepsilon}\right)\left(\left(\log \frac{1}{\varepsilon}\right) m(S_\varepsilon^K(0, t)) - \pi t\right)$$

$$= \pi t\left(\frac{\log \lambda - \log 2}{2} + \gamma + R(K)\right) - \pi^2 T^2(t). \tag{1.e}$$

This result is already proved in [11, 13]. For general $n$, the different terms which appear in (1.c) can be interpreted as follows. For every $k \geqslant 2$, $k$-multiple self-intersections give rise to a corrective term of order $(h_\varepsilon^K)^{-k}$. This corresponds to the fact that the area of the intersection of $k$ independent Wiener sausages of radius $\varepsilon$ is of order $(\log 1/\varepsilon)^{-k}$ (see [12]).

Asymptotic properties of the Wiener sausage have been recently studied by several authors. The limiting result (1.d) and its analogues in higher dimensions have been extended by Chavel and Feldman [3, 4] to Brownian motion on manifolds, and by Sznitman [22] to elliptic diffusions in $\mathbb{R}^d$. The paper [13] contains analogues of (1.e) in higher dimensions ($d \geqslant 3$); in contrast with the case $d=2$, the convergence is only in distribution and the limiting variable is normal. Related results can be found in Weinryb [23]. Finally, Chavel, Feldman, and Rosen [5] have extended (1.e) to Brownian motion on surfaces. In connection with this result, one may ask whether the asymptotic decomposition (1.c) remains true in this differential geometry setting. A construction of renormalized self-intersection local times valid for Brownian motion on manifolds has been given by Le Jan [16].

The paper is organized as follows. Section 2 contains a number of notations and preliminary estimates. In Section 3, we construct the random fields $T^k$ by proving a general version of the convergence (1.a). Although largely inspired by Dynkin's papers [7–9], our method is different from his and adapted to our applications. In particular, we avoid the combinatorial part which plays an important role in Dynkin's approach. On the other hand, we only recover a special case of Dynkin's functionals, namely those introduced in [7]. In Section 4, we establish the basic estimate (1.b) which relates the Wiener sausage to the random fields $T^k$. The proof depends on both the preliminary estimates of Section 2 and the particular construction of the fields $T^k$ given in Section 3. Finally, in Section 5 we investigate results concerning Brownian motion stopped at a constant time and we prove (1.c). We also develop an application of (1.c) to a theorem of Spitzer [21].

Some of the results of the present paper have been announced in [14].

## 2. Notations and Preliminaries

(2.1) Throughout this work, it will be convenient to use the canonical space of planar Brownian motion killed at an independent exponential time $\zeta$. We denote by $\Omega_0 = C(\mathbb{R}_+, \mathbb{R}^2)$ the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^2$ and we set $\Omega = \Omega_0 \times \mathbb{R}_+$. For every $y \in \mathbb{R}^2$, we define the probability $P_y$ on $\Omega$ by

$$P_y = W_y \otimes \xi,$$

where $W_y$ is the Wiener measure with starting point $y$ on $\Omega_0$, and $\xi(dt) = \lambda e^{-\lambda t}\,dt$, for some fixed $\lambda > 0$. For $(\omega, t) \in \Omega$, we define

$$B_s(\omega, t) = \omega(s) \qquad (s \geqslant 0)$$

$$\zeta(\omega, t) = t,$$

so that, under $P_y$, $B = (B_s; s \geqslant 0)$ is a Brownian motion starting at $y$ and $\zeta$ is an independent exponential time. We simply write $P$ for $P_0$, $W$ for $W_0$, and, unless otherwise indicated, we always assume that we are working under the probability $P$. The canonical filtration of $B$ is denoted by $(\mathscr{F}_t; t \geqslant 0)$. We shall sometimes use the canonical filtration of the process $B$ killed at $\zeta$, which is denoted by $(\mathscr{G}_t; t \geqslant 0)$.

For every $t \geqslant 0$, the shift $\theta_t$ is defined on the subset $\Omega^t = \{(\omega, u); u > t\}$ of $\Omega$ by

$$\theta_t(\omega, u) = (\omega_t, u - t), \qquad \omega_t(s) = \omega(s + t) \qquad (s \geqslant 0).$$

We denote by $G$ the Green function of the process $B$ killed at $\zeta$:

$$G(x) = \int_0^\infty e^{-\lambda t}\, p_t(x)\, dt \qquad (x \in \mathbb{R}^2),$$

where $p_t(x)$ is the two-dimensional Brownian density. It is well known (see, e.g., [10, p. 233]) that

$$G(x) = \frac{1}{\pi} K_0(\sqrt{2\lambda}\,|x|), \tag{2.a}$$

where $K_0$ is the usual modified Bessel function. In particular, as $|x| \to 0$,

$$G(x) = \frac{1}{\pi} \log \frac{1}{|x|} + \frac{1}{\pi}\left(\frac{\log 2 - \log \lambda}{2} - \gamma\right) + O\left(|x|^2 \log \frac{1}{|x|}\right), \tag{2.b}$$

where $\gamma$ denotes Euler's constant. Moreover, if $A > 0$ is small enough, we have

$$G(x) = O(e^{-A|x|}) \qquad \text{as} \quad |x| \to \infty.$$

These bounds show that, for every $0<p<\infty$, the function $G(x)^p$ is integrable.

The following property of $G$ will play a basic role in the proof of our main estimates. We may choose two positive constants $C$, $C'$ such that, if

$$\rho(r)=C\left(\frac{1}{r}I(r<1)+\exp-C'r\right) \qquad (r>0), \tag{2.c}$$

we have, whenever $|x|\geqslant 2\,|w|>0$,

$$|G(x+w)-G(x)|\leqslant |w|\;\rho(|x|). \tag{2.d}$$

The bound (2.d) easily follows from well-known properties of the function $K_0$. Observe that $x\to\rho(|x|)$ is integrable with respect to Lebesgue measure on $\mathbb{R}^2$.

We define a random field to be a linear mapping from the space $\mathscr{B}(\mathbb{R}^2,\mathbb{R})$ of bounded measurable functions on $\mathbb{R}^2$ into

$$\mathscr{L}(\Omega, P)=\bigcap_{1\leqslant p<\infty} L^p(\Omega, P).$$

We shall also consider random fields with values in $\mathscr{L}(\Omega_0, W)$.

(2.2) Let $K$ be a compact subset of $\mathbb{R}^2$ with positive logarithmic capacity. Set

$$T_K=\inf\{t\geqslant 0;\,B_t\in K\}.$$

Then, except when $x$ is an irregular point of $\partial K$, we have

$$P_x[T_K<\zeta]=\int\mu_K(dy)\,G(y-x), \tag{2.e}$$

where $\mu_K$ denotes the equilibrium measure of $K$, for Brownian motion killed at time $\zeta$ (see Blumenthal and Getoor [2, p. 285]). The measure $\mu_K$ is supported on $\partial K$ and its total mass $\bar{\mu}_K=\operatorname{cap}(K)$ satisfies

$$\bar{\mu}_K^{-1}=\inf\left\{\iint_{K\times K}\mu(dx)\,\mu(dy)\,G(y-x)\right\},$$

where the infimum is over all probability measures $\mu$ supported on $K$. We now replace $K$ by $\varepsilon K$, $\varepsilon>0$. It easily follows from (2.b) that, as $\varepsilon\to 0$,

$$\bar{\mu}_{\varepsilon K}^{-1}=\frac{1}{\pi}\log\frac{1}{\varepsilon}+\frac{1}{\pi}\left(\frac{\log 2-\log\lambda}{2}-\gamma-R(K)\right)+O\left(\varepsilon^2\log\frac{1}{\varepsilon}\right), \tag{2.f}$$

where

$$R(K) = -\inf\left\{\iint_{K\times K} \mu(dx)\,\mu(dy)\log\frac{1}{|y-x|}\right\} \tag{2.g}$$

is the logarithm of the logarithmic capacity of $K$ ($R(K)$ coincides with $-\pi$ times Robin's constant of $K$, defined as in [18, p. 76]).

For $y\in\mathbb{R}^2$ we write $T_K(y) = T_{y-K}$. The Wiener sausage of radius $\varepsilon$ associated with $B$ and the compact set $K$ on the time interval $[0; t]$ is then defined by

$$S_\varepsilon^K(0, t) = \{y; T_{\varepsilon K}(y) \leqslant t\} = \bigcup_{0\leqslant s\leqslant t} (B_s + \varepsilon K).$$

Similarly for $s\leqslant t$ we set

$$S_\varepsilon^K(s, t) = \bigcup_{s\leqslant u\leqslant t} (B_u + \varepsilon K).$$

From now on we set $h_\varepsilon^K = -(\bar{\mu}_{\varepsilon K})^{-1}$. The following lemma is a straightforward consequence of formulas (2.e) and (2.d).

LEMMA 2.1. *Suppose that $K$ is contained in the closed unit disk $D$. There exist two positive constants $C_1, C_2$ such that, whenever $\varepsilon\in(0,\frac{1}{2})$ and $w, x, y\in\mathbb{R}^2$ satisfy $|w|\leqslant\varepsilon$, $|x|\geqslant 4\varepsilon$, $|y|\geqslant 4\varepsilon$, and $|x-y|\geqslant 4\varepsilon$, we have*

(i) $$\left(\log\frac{1}{\varepsilon}\right) P_w[T_{\varepsilon K}(x) < \zeta] \leqslant C_1 G\left(\frac{x}{2}\right)$$

(ii) $$\left(\log\frac{1}{\varepsilon}\right)^2 P_w[T_{\varepsilon K}(x) < \zeta;\ T_{\varepsilon K}(y) < \zeta] \leqslant C_2\left(G\left(\frac{x}{2}\right) + G\left(\frac{y}{2}\right)\right) G\left(\frac{y-x}{2}\right) \tag{2.h}$$

(iii) $$|h_\varepsilon^K P_{x+w}[T_{\varepsilon K}(y) < \zeta] + G(y-x)| \leqslant 2\varepsilon\rho(|y-x|),$$

*where $\rho$ is defined in* (2.c).

In proving part (ii) of the lemma we consider the two cases $(T_{\varepsilon K}(x)\leqslant T_{\varepsilon K}(y))$ and $(T_{\varepsilon K}(y)\leqslant T_{\varepsilon K}(x))$ and we apply the strong Markov property at $T_{\varepsilon K}(x)$, resp. $T_{\varepsilon K}(y)$. Details are left to the reader.

(2.3) We now recall a few basic facts about additive functionals. It will be convenient to consider additive functionals with finite variation, i.e., not necessarily nondecreasing. An additive functional $A=(A_t, t\geqslant 0)$ of the process $B$ is integrable if

$$\int dx\, E_x\left[\int_0^t |A(ds)|\right]<\infty$$

for some (and hence for every) $t>0$. With each integrable additive functional $A=(A_t, t\geqslant 0)$ we can associate a unique (signed) measure $\nu$ on $\mathbb{R}^2$ such that

$$\int dx\, E_x\left[\int_0^t I(B_s\in H)\, A(ds)\right]=t\nu(H),$$

for every $t\geqslant 0$, $H$ measurable subset of $\mathbb{R}^2$ (see [1]). Let $|\nu|$ denote the total variation of $\nu$. If the $\lambda$-potential of $|\nu|$ is bounded, we also have, for every $x\in\mathbb{R}^2$,

$$E_x[A(\zeta)]=G\nu(x)=\int \nu(dy)\, G(x-y). \tag{2.i}$$

Observe that $\text{supp}(dA)\subset\{s;\, B_s\in\text{supp}(\nu)\}$.

Let $\mathscr{A}$ denote the set of all integrable additive functionals of $B$ such that the associated measure $\nu$ is supported on the closed unit disk $D$ and the $\lambda$-potential of $|\nu|$ is bounded. We denote by $\mathscr{A}_1$ the subset of $\mathscr{A}$ which consists of all $A$ such that $\nu$ is a probability measure.

Let $A\in\mathscr{A}$. For every $y\in\mathbb{R}^2$, $\varepsilon>0$, we define a new additive functional $A_\varepsilon^y$ by requiring $A_\varepsilon^y$ to be associated with the measure $\nu_\varepsilon^y$ defined by

$$\int \nu_\varepsilon^y(dw)\, f(w)=\int \nu(dw)\, f(y+\varepsilon w).$$

Note that $\nu_\varepsilon^y$ is supported on the closed disk of center $y$ and radius $\varepsilon$. It follows from (2.b) that the $\lambda$-potential of $|\nu_\varepsilon^y|$ is bounded by $C(\log 1/\varepsilon)$ for some constant $C$ independent of $y\in\mathbb{R}^2$, $\varepsilon\in(0,\frac{1}{2})$. Standard arguments then give the bound: for every $k\geqslant 1$,

$$\sup\{E_x[|A_\varepsilon^y|^k],\, x\in\mathbb{R}^2\}\leqslant C_k(\log 1/\varepsilon)^k.$$

Suppose that $A\in\mathscr{A}_1$. It easily follows from (2.d) and (2.i) that, for $\varepsilon>0$ and $x$, $y$, $w$ such that $|y-x|\geqslant 4\varepsilon$ and $|w|\leqslant\varepsilon$,

$$|E_{x+w}[A_\varepsilon^y(\zeta)]-G(y-x)|\leqslant 2\varepsilon\rho(|y-x|). \tag{2.j}$$

(2.4) For every integer $k\geqslant 1$ and for $t\geqslant 0$, set

$$D_k(t)=\{(s_1,...,s_k);\, 0\leqslant s_1\leqslant\cdots\leqslant s_k\leqslant t\}.$$

The set $D_k(\zeta)$ will be denoted by $D_k$.

Let $\delta_{(s)}(\cdot)$ denote the Dirac measure at $s$. Let $\alpha_1, ..., \alpha_k$ be $k$ finite measures on $[0; t]$ and $g_2, ..., g_k$ $k-1$ bounded measurable functions on $[0; t]$. The formula

$$\gamma_k(ds_1 \cdots ds_k) = \alpha_1(ds_1)(\alpha_2(ds_2) - g_2(s_1)\,\delta_{(s_1)}(ds_2)) \cdots (\alpha_k(ds_k) - g_k(s_{k-1})\,\delta_{(s_{k-1})}(ds_k))$$

gives a perfectly well-defined measure on $D_k(t)$. Suppose in addition that $\alpha_1, ..., \alpha_k$ are supported on a closed set $H$. Then $\gamma_k(\cdot)$ is supported on $\{(s_1, ..., s_k); s_i \in H, \text{ for } i = 1, ..., k\}$.

Let $\gamma'_j\ (dt_1, ..., dt_j)$ be another measure of the same type, now defined on $D_j(t)$:

$$\gamma'_j(dt_1 \cdots dt_j) = \alpha'_1(dt_1)(\alpha'_2(dt_2) - g'_2(t_1)\,\delta_{(t_1)}(dt_2)) \cdots (\alpha'_j(dt_j) - g'_j(t_{j-1})\,\delta_{(t_{j-1})}(dt_j)).$$

Suppose that $\alpha'_1, ..., \alpha'_j$ are supported on a closed set $H'$ and that $H$ and $H'$ are disjoint. Consider the product measure $\gamma_k \otimes \gamma'_j$, which is defined on $D_k(t) \times D_j(t)$. By considering the different possible orderings of $(s_1, ..., s_k, t_1, ..., t_j)$, we can interpret $\gamma_k \otimes \gamma'_j$ as a sum of measures of the same type as $\gamma_k$. The disjointness of $H$ and $H'$ is here essential. Let us illustrate this by an example. Suppose that $j = k$; then, on the set

$$\{s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant \cdots \leqslant s_k \leqslant t_k\}$$

the measure $\gamma_k \otimes \psi'_k$ coincides with

$$\alpha_1(ds_1)\,\alpha'_1(dt_1)\,\alpha_2(ds_2)\,\alpha'_2(dt_2) \cdots \alpha_k(ds_k)\,\alpha'_k(dt_k),$$

that is, all the Dirac measures $\delta_{(s_{i-1})}(ds_i)$, $\delta_{(t_{i-1})}(dt_i)$ can be dropped. The point is that, since $H$ and $H'$ are disjoint, a set of the type $\{s_i = t_{i'}\}$ has zero measure for $\gamma_k \otimes \gamma'_j$. Similar statements obviously apply to the product of $p$ measures of the same type as $\gamma_k$.

(2.5) The following elementary lemma plays a basic role in this work.

LEMMA 2.2. *Let $A \in \mathscr{A}_1$, $U$ a continuous $\mathscr{G}_t$-predictable process with finite variation, and $V$ a measurable function on $\Omega$ such that*

$$\sup\{E_z[|V|]; z \in \mathbb{R}^2\} < \infty.$$

*Set*

$$U^*(t) = \int_0^t |U(ds)|.$$

*Suppose that, for some* $y \in \mathbb{R}^2$, $\delta > 0$, *and for every* $x \in \mathbb{R}^2$

(i) $\int_0^\zeta A(ds)\, I(|B_s - y| > \delta) = 0 \quad P_x$ *a.s.*,

(ii) $\int_0^\zeta U^*(ds)\, I(|B_s - y| > \delta) = 0 \quad P_x$ *a.s.*

*Then, for every* $x \in \mathbb{R}^2$,

$$\left| E_x \left[ \int_0^\zeta U(ds) \int_s^\zeta (A(dt) - E_{B_s}[A(\zeta)]\, \delta_{(s)}(dt))\, V \circ \theta_t \right] \right|$$
$$\leqslant 2E_x[U^*(\zeta)] \sup\{E_x[A(\zeta)];\, |z-y| \leqslant \delta\}\, \omega_\delta(y, V),$$

*where*

$$\omega_\delta(y, V) = \sup\{|E_z[V] - E_y[V]|;\, |z-y| \leqslant \delta\}.$$

*Proof.* We first observe that, by the definition of an additive functional, we have on the set $\{s < \zeta\}$

$$\int_s^\zeta (A(dt) - E_{B_s}[A(\zeta)]\, \delta_{(s)}(dt))\, V \circ \theta_t = \psi \circ \theta_s,$$

where

$$\psi = \int_0^\zeta (A(dt) - E_{B_0}[A(\zeta)]\, \delta_{(0)}(dt))\, V \circ \theta_t$$
$$= \int_0^\zeta A(dt)\, V \circ \theta_t - E_{B_0}[A(\zeta)]\, V.$$

Then,

$$E_x \left[ \int_0^\zeta U(ds)\, \psi \circ \theta_s \right] = E_x \left[ \int_0^\zeta U(ds)\, E_{B_s}[\psi] \right], \tag{2.k}$$

where we have simply replaced the proces $I(s < \zeta)\, \psi \circ \theta_s$ by its $\mathscr{G}_t$-predictable projection $I(s < \zeta)\, E_{B_s}[\psi]$. Similarly, for $z \in \mathbb{R}^2$ such that $|z - y| \leqslant \delta$,

$$|E_z[\psi]| = \left| E_z \left[ \int_0^\zeta A(dt)\, V \circ \theta_t \right] - E_z[A(\zeta)]\, E_z[V] \right|$$
$$= \left| E_z \left[ \int_0^\zeta A(dt)\, E_{B_t}[V] \right] - E_z[A(\zeta)]\, E_z[V] \right|$$
$$\leqslant 2E_z[A(\zeta)]\, \omega_\delta(y, V), \tag{2.l}$$

using assumption (i). The lemma follows from (2.k), (2.l), and assumption (ii). ∎

Note that condition (ii) of the lemma is satisfied if and only if the measure associated with $A$ is supported on the disk of center $y$ and radius $\delta$. The goal of the next two subsections is to bound $\omega_\delta(y, V)$ for various functionals $V$.

(2.6) PROPOSITION 2.3. *Let $p \geq 1$ and $A^1, ..., A^p \in \mathscr{A}_1$. Let $\varepsilon_1, ..., \varepsilon_p > 0$ and $y_0, y_1, ..., y_p \in \mathbb{R}^2$. Set*

$$V_{\varepsilon_1, ..., \varepsilon_p}(y_1, ..., y_p) = \int_{D_p} A_{\varepsilon_1}^{1, y_1}(dt_1) \cdots A_{\varepsilon_p}^{p, y_p}(dt_p),$$

*and $\varepsilon^* = \max\{\varepsilon_i, i = 1, ..., p\}$. Assume that $\varepsilon^* < \frac{1}{2}$ and $|y_i - y_{i+1}| > 4\varepsilon^*$ for $i = 0, 1, ..., p-1$. Then,*

$$\left| E_{y_0}[V_{\varepsilon_1, ..., \varepsilon_p}(y_1, ..., y_p)] - \prod_{i=1}^{p} G(y_i - y_{i-1}) \right|$$
$$\leq 2\varepsilon^* \sum_{j=1}^{p} \rho(|y_j - y_{j-1}|) \prod_{\substack{i=1 \\ i \neq j}}^{p} G\left(\frac{y_i - y_{i-1}}{2}\right).$$

*Proof.* Let $p \geq 2$. Then,

$$E_{y_0}\left[\int_{D_p} A_{\varepsilon_1}^{1, y_1}(dt_1) \cdots A_{\varepsilon_p}^{p, y_p}(dt_p)\right]$$
$$= E_{y_0}\left[\int_{D_{p-1}} A_{\varepsilon_1}^{1, y_1}(dt_1) \cdots A_{\varepsilon_{p-1}}^{p-1, y_{p-1}}(dt_{p-1})\, E_{B_{t_{p-1}}}[A_{\varepsilon_p}^{p, y_p}(\zeta)]\right],$$

where we have simply replaced $I(t_{p-1} < \zeta) \int_{t_{p-1}}^{\zeta} A_{\varepsilon_p}^{p, y_p}(dt_p)$ by its predictable projection $I(t_{p-1} < \zeta)\, E_{B_{t_{p-1}}}[A_{\varepsilon_p}^{p, y_p}(\zeta)]$. If $y$ is such that $|y - y_{p-1}| < \varepsilon^*$ we have by (2.j)

$$|E_y[A_{\varepsilon_p}^{p, y_p}(\zeta)] - G(y_p - y_{p-1})| \leq 2\varepsilon^* \rho(|y_p - y_{p-1}|).$$

It follows that

$$\left| E_{y_0}\left[\int_{D_p} A_{\varepsilon_1}^{1, y_1}(dt_1) \cdots A_{\varepsilon_p}^{p, y_p}(dt_p) \right.\right.$$
$$\left.\left. - G(y_p - y_{p-1}) \int_{D_{p-1}} A_{\varepsilon_1}^{1, y_1}(dt_1) \cdots A_{\varepsilon_{p-1}}^{p-1, y_{p-1}}(dt_{p-1})\right]\right|$$
$$\leq 2\varepsilon^* \rho(|y_p - y_{p-1}|)\, E_{y_0}\left[\int_{D_{p-1}} A_{\varepsilon_1}^{1, y_1}(dt_1) \cdots A_{\varepsilon_{p-1}}^{p-1, y_{p-1}}(dt_{p-1})\right]$$
$$\leq 2\varepsilon^* \rho(|y_p - y_{p-1}|) \prod_{i=2}^{p-1} G\left(\frac{y_i - y_{i-1}}{2}\right),$$

by (2.l). The proof is now easily completed by induction. ∎

(2.7) PROPOSITION 2.4. *Let $A \in \mathcal{A}_1$ and $K$ a compact subset of $D$ with positive capacity. Let $n \geq 1$ and $n' = n$ or $n-1$. For $\varepsilon > 0$, $y$, $z \in \mathbb{R}^2$, we consider the functional $V_\varepsilon(y, z)$ defined by*

$$V_\varepsilon(y, z) = (-h_\varepsilon^K)^{n'} \int_{D_n} \prod_{i=1}^{n} A_\varepsilon^y(dt_i) \prod_{i=1}^{n'} I(z \in S_\varepsilon^K(t_i, t_{i+1}))$$

*or, alternatively,*

$$V_\varepsilon(y, z) = (-h_\varepsilon^K)^{n} \int_{D_{n'}} \prod_{i=1}^{n'} A_\varepsilon^z(dt_i) \prod_{i=1}^{n} I(y \in S_\varepsilon^K(t_{i-1}, t_i)),$$

*where, by convention, $t_0 = 0$, $t_{n+1} = \zeta$ if $n' = n$ in the first case, $t_n = \zeta$ if $n' = n-1$ in the second case.*

*Then, for every $\varepsilon \in (0, \frac{1}{2})$ and $x, y, z \in \mathbb{R}^2$ such that $|y-x| > 2\varepsilon$, $|z-y| > 4\varepsilon$, we have*

$$\begin{aligned} &|E_x[V_\varepsilon(y, z)] - G(y-x)\, G(z-y)^{n+n'-1}| \\ &\leq 2\varepsilon \left( (n+n'-1)\, G\left(\frac{y-x}{2}\right) G\left(\frac{z-y}{2}\right)^{n+n'-2} \rho(|z-y|) \right. \\ &\qquad \left. + \rho(|y-x|)\, G\left(\frac{z-y}{2}\right)^{n+n'-1} \right). \end{aligned}$$

*Remark.* It immediately follows from Proposition 2.4 that, whenever $|y-x| > 4\varepsilon$, $|z-y| > 4\varepsilon$,

$$\omega_\varepsilon(x, V_\varepsilon(y, z)) \leq \varepsilon f(y-x, z-y),$$

for some integrable function $f(y, z)$ whose exact form will be unimportant. To be specific, this bound holds except in the trivial case $n = 1$, $n' = 0$, which gives $V_\varepsilon(y, z) = V_\varepsilon(y) = A_\varepsilon^y(\zeta)$ or $(-h_\varepsilon^K)\, I(y \in S_\varepsilon^K(0, \zeta))$. In both cases, we simply have

$$\omega_\varepsilon(x, V_\varepsilon(y)) \leq 6\varepsilon\rho(|y-x|).$$

*Proof.* We first observe that, when $|y-x| > 2\varepsilon$ and $|z-y| > 4\varepsilon$,

$$E_x[V_\varepsilon(y, z)] \leq G\left(\frac{y-x}{2}\right) G\left(\frac{z-y}{2}\right)^{n+n'-1}. \tag{2.m}$$

This bound is an easy consequence of the formulas

$$E_x[A_\varepsilon^y(\zeta)] = \int v_\varepsilon^y(dw)\, G(w-x),$$

$$P_x[y \in S_\varepsilon^K(0, \zeta)] = \int \mu_{\varepsilon K}(dw)\, G(y-w-x), \qquad -h_\varepsilon^K = \bar{\mu}_{\varepsilon K}^{-1}.$$

Here $\nu$ is the measure associated with $A$, $\nu_\varepsilon^y$ is defined in subsection (2.3), and $\mu_{\varepsilon K}$ in subsection (2.2).

We only treat the case when $n' = n \geqslant 2$ and $V_\varepsilon(y, z)$ is defined by the first formula. Then

$$\begin{aligned}
E_x[V_\varepsilon(y,z)] &= E_x\left[(-h_\varepsilon^K)^n \int_{D_n} \prod_{i=1}^{n} A_\varepsilon^y(dt_i) \prod_{i=1}^{n} I(z \in S_\varepsilon^K(t_i, t_{i+1}))\right] \\
&= E_x\left[(-h_\varepsilon^K)^{n-1} \int_{D_n} \prod_{i=1}^{n} A_\varepsilon^y(dt_i) \prod_{i=1}^{n-1} I(z \in S_\varepsilon^K(t_i, t_{i+1}))\right. \\
&\qquad \left. \times (-h_\varepsilon^K)\, P_{B_{t_n}}[z \in S_\varepsilon^K(0, \zeta)]\right].
\end{aligned}$$

Next we use Lemma 2.1, which shows that, if $|w - y| \leqslant \varepsilon$,

$$|h_\varepsilon^K P_w[z \in S_\varepsilon^K(0, \zeta)] + G(z - y)| \leqslant 2\varepsilon\rho(|z - y|).$$

Taking (2.m) into account, it follows that

$$\begin{aligned}
&\Bigg| E_x[V_\varepsilon(y,z)] \\
&\quad - G(z-y)\, E_x\left[(-h_\varepsilon^K)^{n-1} \int_{D_n} \prod_{i=1}^{n} A_\varepsilon^y(dt_i) \prod_{i=1}^{n-1} I(z \in S_\varepsilon^K(t_i, t_{i+1}))\right]\Bigg| \\
&\leqslant 2\varepsilon G\left(\frac{y-x}{2}\right) G\left(\frac{z-y}{2}\right)^{2n-1} \rho(|z-y|).
\end{aligned} \tag{2.n}$$

Set

$$T_\varepsilon(z, t) = \inf\{s > t;\, B_s \in z - \varepsilon K\}.$$

Then,

$$\begin{aligned}
&E_x\left[\int_{D_n} \prod_{i=1}^{n} A_\varepsilon^y(dt_i) \prod_{i=1}^{n-1} I(z \in S_\varepsilon^K(t_i, t_{i+1}))\right] \\
&\quad = E_x\left[\int_{D_{n-1}} \prod_{i=1}^{n-1} A_\varepsilon^y(dt_i) \prod_{i=1}^{n-1} I(z \in S_\varepsilon^K(t_i, t_{i+1})) \int_{T_\varepsilon(z, t_{n-1})}^{\zeta} A_\varepsilon^y(dt_n)\right] \\
&\quad = E_x\left[\int_{D_{n-1}} \prod_{i=1}^{n-1} A_\varepsilon^y(dt_i) \prod_{i=1}^{n-1} I(z \in S_\varepsilon^K(t_i, t_{i+1}))\, E_{B_{T_\varepsilon(z, t_{n-1})}}[A_\varepsilon^y(\zeta)]\right].
\end{aligned}$$

By (2.j) we have, if $|w - z| \leqslant \varepsilon$,

$$|E_w[A_\varepsilon^y(\zeta)] - G(y - z)| \leqslant 2\varepsilon\rho(|y - z|).$$

Coming back to (2.n), we finally obtain

$$\Big| E_x[V_\varepsilon(y,z)] - G(z-y)^2 E_x\Big[(-h_\varepsilon^K)^{n-1}\int_{D_{n-1}} \prod_{i=1}^{n-1} A_\varepsilon^y(dt_i) \prod_{i=1}^{n-1} I(z\in S_\varepsilon^K(t_i,t_{i+1}))\Big]\Big|$$
$$\leqslant 4\varepsilon G\left(\frac{y-x}{2}\right) G\left(\frac{z-y}{2}\right)^{2n-1} \rho(|z-y|).$$

The proof is now easily completed by induction. ∎

## 3. Functionals Associated with Self-Intersections

(3.1) The main goal of this section is to construct, for every integer $k\geqslant 1$, a random field $\varphi \to T^k\varphi$ associated with $k$-multiple points of the Brownian motion $B$. We will also prove that these random fields coincide with those introduced by Dynkin [7]. However, our construction is different from Dynkin's and yields new approximations of $T^k\varphi$ which may have some independent interest. Moreover, the basic lemma needed in the proof of Theorem 3.1 will play an important role in the next sections, in our study of asymptotics of the planar Wiener sausage.

We keep the notations introduced in the previous section. In particular, with each additive functional $A\in\mathscr{A}$ we associate a family $(A_\varepsilon^y;\, y\in\mathbb{R}^2,\ \varepsilon>0)$ of new additive functionals obtained by translating and rescaling $A$. For $\varepsilon>0$, the function $h_\varepsilon^A$ is defined by

$$h_\varepsilon^A(y) = -E_0[A_\varepsilon^y(\zeta)] = -\int G(y+\varepsilon w)\, \nu_A(dw),$$

where $\nu_A$ is the measure associated with $A$.

We also need the following additional notation. We consider $p$ integers $k_1, k_2, ..., k_p \geqslant 1$. We denote by $\Gamma^*(k_1, ..., k_p)$ the set of all mappings $\sigma$ from $\{1, 2, ..., k_1+k_2+\cdots+k_p\}$ onto $\{1, ..., p\}$, such that $|\sigma^{-1}(i)| = k_i$, for $i=1, ..., p$. We shall also use the set $\Gamma(k_1, ..., k_p)$ of all mappings $\sigma\in\Gamma^*(k_1, ..., k_p)$ such that $\sigma(n)\neq\sigma(n-1)$ for $n=2, 3, ..., k_1+\cdots+k_p$. Note that $\Gamma(k_1, ..., k_p)$ may be empty: it is so, e.g., if $p=1$, $k_1\geqslant 2$.

(3.2) Theorem 3.1. *For every integer $k\geqslant 1$, there exists a unique random field $\varphi\to T^k\varphi$, defined for $\varphi\in\mathscr{B}(\mathbb{R}^2,\mathbb{R})$ and taking values in $\mathscr{L}(\Omega, P)$, which satisfies the following properties. Let $\varphi\in\mathscr{B}(\mathbb{R}^2,\mathbb{R}_+)$ and $A^1, ..., A^k\in\mathscr{A}_1$. For $\varepsilon>0$, set*

$$T_\varepsilon^k\varphi = \int dy\, \varphi(y) \int_{D_k} A_\varepsilon^{1,y}(dt_1) \prod_{i=2}^{n} (A_\varepsilon^{i,y}(dt_i) + h_\varepsilon^{A^i}(y-B_{t_{i-1}})\, \delta_{(t_{i-1})}(dt_i)).$$

*Then, for every integer* $p \geqslant 1$,

$$E[(T_\varepsilon^k \varphi - T^k \varphi)^{2p}] \leqslant C_p \, \|\varphi\|_\infty^{2p} \, \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{2pk},$$

*where the constant* $C_p$ *does not depend on* $\varphi$, $\varepsilon$. *Moreover, for* $k_1, ..., k_p \geqslant 1$, $\varphi_1, ..., \varphi_p \in \mathscr{B}(\mathbb{R}^2, \mathbb{R}_+)$,

$$E[T^{k_1}\varphi_1 \cdots T^{k_p}\varphi_p] = \int dy_1 \cdots dy_p \, \varphi_1(y_1) \cdots \varphi_p(y_p)$$

$$\times \sum_{\sigma \in \Gamma(k_1, ..., k_p)} \prod_{i=1}^{k_1 + \cdots + k_p} G(y_{\sigma(i)} - y_{\sigma(i-1)})$$

*with the usual convention* $y_{\sigma(0)} = 0$.

*Remarks.* (i) Take $A^1 = A^2 = \cdots = A^p = l$, the additive functional associated with the uniform probability measure on the circle of center 0 and radius $\varepsilon$. Then $h_\varepsilon^l(z) = h_\varepsilon$ does not depend on $z$ such that $|z| = \varepsilon$. It follows that, in this special case,

$$\begin{aligned} T_\varepsilon^k \varphi &= \int dy \, \varphi(y) \int_{D_k} l_\varepsilon^y(dt_1) \prod_{i=2}^{k} (l_\varepsilon^y(dt_i) + h_\varepsilon \, \delta_{(t_{i-1})}(dt_i)) \\ &= \int dy \, \varphi(y) \sum_{i=2}^{k} \binom{k-1}{i-1} \frac{h_\varepsilon^{k-i}}{i!} (l_\varepsilon^y)^i. \end{aligned}$$

This special approximation of $T^k \varphi$ was used in [14]. It is easily seen that

$$\begin{aligned} h_\varepsilon &= -G(\varepsilon) + O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right) \\ &= -\frac{1}{\pi} \log \frac{1}{\varepsilon} - \frac{1}{\pi}\left(\frac{\log 2 - \log \lambda}{2} - \gamma\right) + O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right). \end{aligned}$$

(ii) Theorem 3.1 can be slightly extended as follows. Suppose that $A^1, ..., A^k$ belong to $\mathscr{A}$ instead of $\mathscr{A}_1$. For $i = 1, ..., k$, let $v^i$ be the measure on $\mathbb{R}^2$ associated with $A^i$. Then,

$$E[(T_\varepsilon^k \varphi - \bar{v}^1 \cdots \bar{v}^k T^k \varphi)^{2p}] \leqslant C_p \, \|\varphi\|_\infty^2 \, \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{2pk},$$

where $\bar{v}^i = v^i(\mathbb{R}^2)$. This result is easily deduced from Theorem 3.1 by linearity arguments.

The proof of Theorem 3.1 depends on the following basic lemma. The notation $L^r((\mathbb{R}^2)^p, \mathbb{R}_+)$ refers to the set of all nonnegative measurable functions on $(\mathbb{R}^2)^p$ whose $r$th power is integrable.

LEMMA 3.2. *Suppose we are given* $k_1, \ldots, k_p \geqslant 1$ *and a family* $(A_j^i;\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant k_i)$ *of elements of* $\mathscr{A}_1$. *For every* $i = 1, \ldots, p$, $\varepsilon > 0$, *and* $y \in \mathbb{R}^2$, *set*

$$X_\varepsilon^i(y) = \int_{D_{k_i}} A_{1,\varepsilon}^{i,y}(dt_1) \prod_{j=2}^{k_i} \left(A_{j,\varepsilon}^{i,y}(dt_j) + h_\varepsilon^{A_j^i}(y - B_{t_{j-1}})\, \delta_{(t_{j-1})}(dt_j)\right).$$

*Then there exists a function* $F \in L^1((\mathbb{R}^2)^p, \mathbb{R}_+)$ *such that, for any* $\varepsilon \in (0, \frac{1}{2})$, $\varepsilon_1, \ldots, \varepsilon_p \in [\varepsilon/2, \varepsilon]$, *and* $y_1, \ldots, y_p$ *distinct points in* $\mathbb{R}^2 - \{0\}$,

$$\left| E\left[\prod_{i=1}^{p} X_{\varepsilon_i}^i(y_i)\right] - \sum_{\sigma \in \Gamma(k_1, \ldots, k_p)} \prod_{i=1}^{k_1 + \cdots + k_p} G(y_{\sigma(i)} - y_{\sigma(i-1)}) \right|$$
$$\leqslant \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{k_1 + \cdots + k_p} F(y_1, \ldots, y_p).$$

*Proof.* We first need to obtain some preliminary bounds. For $y \in \mathbb{R}^2$, set

$$T_\varepsilon(y) = \inf\{t \geqslant 0;\ |B_t - y| \leqslant \varepsilon\}.$$

For $A \in \mathscr{A}_1$ and $k \geqslant 1$ we have, by the strong Markov property and Lemma 2.1,

$$E[(A_\varepsilon^y(\zeta))^k] \leqslant P[T_\varepsilon(y) \leqslant \zeta] \sup_{z \in \mathbb{R}^2} E_z[(A_\varepsilon^y(\zeta))^k]$$
$$\leqslant C\left(\log \frac{1}{\varepsilon}\right)^{k-1} \left(G\left(\frac{y}{2}\right) \wedge \log \frac{1}{\varepsilon}\right). \tag{3.a}$$

Note that for every $a > 0$, $y \to G(y)^a$ is integrable. We also have

$$|h_\varepsilon^A(y)| \leqslant C \log \frac{1}{\varepsilon} \qquad \left(y \in \mathbb{R}^2, \varepsilon \in \left(0, \frac{1}{2}\right)\right). \tag{3.b}$$

Expanding the product in the definition of $X_\varepsilon^i(y)$ we deduce from (3.b) that

$$|X_\varepsilon^i(y)| \leqslant C \sum_{k=2}^{k_i} \left(\log \frac{1}{\varepsilon}\right)^{k_i - k} \sum_{1 = j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k \leqslant k_i} \prod_{m=1}^{k} A_{j_m,\varepsilon}^{i,y}(\zeta).$$

Hence, by (3.a) and the Hölder inequality, for $n \geqslant 1$,

$$E[|X_\varepsilon^i(y)|^n] \leqslant C\left(\log \frac{1}{\varepsilon}\right)^{nk_i - 1} \left(G\left(\frac{y}{2}\right) \wedge \log \frac{1}{\varepsilon}\right). \tag{3.c}$$

Here the constant $C$ depends only on $n$, $k_i$, and the additive functionals $A_j^i$

(in fact, only on a bound for the $\lambda$-potentials of the $A_j^i$). It follows from (3.c) that

$$\left| E\left[\prod_{i=1}^{p} X_{\varepsilon_i}^i(y_i)\right]\right| \leqslant C\left(\log\frac{1}{\varepsilon}\right)^{k_1+\cdots+k_p-1} \prod_{i=1}^{p} G\left(\frac{y_i}{2}\right)^{1/p}. \tag{3.d}$$

As a consequence of these bounds, we may assume in the proof of Lemma 3.2 that $|y_i| > 4\varepsilon$ and $|y_i - y_j| > 4\varepsilon$ for $i \neq j$. Indeed, suppose that we have proved the bound of Lemma 3.2 in this special case, with a certain function $F_1$. Take $r \in (1, 2)$ and set

$$f(y_1, ..., y_p) = \left(1 + \sum_{i=1}^{p} |y_i|^{-r} + \sum_{i \neq j} |y_i - y_j|^{-r}\right) \times\left(\prod_{i=1}^{p} G\left(\frac{y_i}{2}\right)^{1/p} + \sum_{\sigma} \prod_{i=1}^{k_1+\cdots+k_p} G(y_{\sigma(i)} - y_{\sigma(i-1)})\right).$$

It follows from (3.d) that the bound of Lemma 3.2 holds with $F = F_1 + Cf$ for some constant $C$.

From now on, we assume $|y_i| > 4\varepsilon$ and $|y_i - y_j| > 4\varepsilon$ for $i \neq j$. We have

$$E\left[\prod_{i=1}^{p} X_{\varepsilon_i}^i(y_i)\right] = E\left[\prod_{i=1}^{p} \int_{D_{k_i}} A_{1,\varepsilon_i}^{i,y_i}(dt_1^i) \prod_{j=2}^{k_i} (A_{j,\varepsilon_i}^{i,y_i}(dt_j^i) + h_{\varepsilon_i}^{i,j}(y_i - B_{t_{j-1}^i})\,\delta_{(t_{j-1}^i)}(dt_j^i))\right],$$

where we set $h_\varepsilon^{i,j} = h_\varepsilon^{A_j^i}$. We can rewrite the above right-hand side as a multiple integral over the product $\prod_i D_{k_i}$, which involves the time variables $t_j^i$ ($i = 1, ..., p$; $j = 1, ..., k_i$),

$$E\left[\prod_{i=1}^{p} X_{\varepsilon_i}^i(y_i)\right] = E\left[\int_{D_{k_1}\times\cdots\times D_{k_p}} \Delta_{\varepsilon_1,...,\varepsilon_p}^{y_1,...,y_p}(dt_j^i;\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant k_i)\right], \tag{3.e}$$

where $\Delta_{\varepsilon_1,...,\varepsilon_p}^{y_1,...,y_p}(dt_j^i;\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant k_i)$ is the signed measure on $\prod_{i=1}^{p} D_{k_i}$ defined by

$$\Delta_{\varepsilon_1,...,\varepsilon_p}^{y_1,...,y_p}(dt_j^i) = \prod_{i=1}^{p} A_{1,\varepsilon_i}^{i,y_i}(dt_1^i) \prod_{j=2}^{k_i} (A_{j,\varepsilon_i}^{i,y_i}(dt_j^i) + h_{\varepsilon_i}^{i,j}(y_i - B_{t_{j-1}^i})\,\delta_{(t_{j-1}^i)}(dt_j^i)).$$

We now consider the different possible orderings of the $t_j^i$, taking into account the fact that we must have $t_j^i \leqslant t_{j'}^i$ if $j \leqslant j'$. Any such ordering will be associated with one (and only one) $\sigma \in \Gamma^*(k_1, ..., k_p)$, in the following way. For any $\sigma \in \Gamma^*(k_1, ..., k_p)$ define

$$\beta_\sigma(n) = |\{m \leqslant n;\ \sigma(m) = \sigma(n)\}| \qquad (n = 1, ..., k_1 + \cdots + k_p).$$

Note that $\beta_\sigma(n) \in \{1, 2, ..., k_{\sigma(n)}\}$. When there is no risk of confusion, we write $\beta$ instead of $\beta_\sigma$. Set

$$\Lambda_\sigma = \left\{ (t^i_j) \in \prod_i D_{k_i} : t^{\sigma(n-1)}_{\beta(n-1)} \leqslant t^{\sigma(n)}_{\beta(n)}, \text{ for } n = 2, 3, ..., k_1 + \cdots + k_p \right\}.$$

It is trivial that

$$\prod_i D_{k_i} = \bigcup_{\sigma \in \Gamma^*(k_1, ..., k_p)} \Lambda_\sigma.$$

This is not a disjoint union. However, if $\sigma \neq \sigma'$, the set $\Lambda_\sigma \cap \Lambda_{\sigma'}$ is contained in a set of the form $\{t^i_j = t^{i'}_{j'}\}$, with $i \neq i'$, and it is easily seen that the measure $\Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}$ gives zero mass to a set of this type. The point is that, since $|y_i - y_{i'}| \geqslant 4\varepsilon$, the additive functionals $A^{i, y_i}_{j, \varepsilon_i}$, $A^{i', y_{i'}}_{j', \varepsilon_{i'}}$ have disjoint supports. It follows from these remarks that

$$E\left[\prod_{i=1}^{p} X^i_{\varepsilon_i}(y_i)\right] = \sum_{\sigma \in \Gamma^*(k_1, ..., k_p)} E\left[\int_{\Lambda_\sigma} \Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}(dt^i_j)\right]. \tag{3.f}$$

*Claim.* If $\sigma \in \Gamma^*(k_1, ..., k_p) - \Gamma(k_1, ..., k_p)$, there exists a function $F_\sigma \in L^1((\mathbb{R}^2)^p, \mathbb{R}_+)$ such that for all $\varepsilon \in (0, \frac{1}{2})$, $\varepsilon_1, ..., \varepsilon_p \in (\varepsilon/2, \varepsilon)$, and $y_1, ..., y_p$ distinct points of $\mathbb{R}^2 - \{0\}$,

$$\left| E\left[\int_{\Lambda_\sigma} \Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}(dt^i_j)\right]\right| \leqslant \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{k_1 + \cdots + k_p} F_\sigma(y_1, ..., y_p).$$

Assume that the claim is proved. Then we can complete the proof of Lemma 3.2 as follows. We first note that for $\sigma \in \Gamma(k_1, ..., k_p)$ we have

$$\begin{aligned} &I((t^i_j) \in \Lambda_\sigma)\, \Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}(dt^i_j) \\ &\quad = I((t^i_j) \in \Lambda_\sigma) \prod_{n=1}^{k_1 + \cdots + k_p} A^{\sigma(n), y_{\sigma(n)}}_{\beta(n), \varepsilon_{\sigma(n)}}(dt^{\sigma(n)}_{\beta(n)}), \end{aligned} \tag{3.g}$$

where $\beta = \beta_\sigma$ as above. The point is that, since $\Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}$ gives zero mass to a set of the type $\{t^i_j = t^{i'}_{j'}\}$ with $i \neq i'$, we may replace, in the left-hand side of (3.g), $I((t^i_j) \in \Lambda_\sigma)$ by

$$\prod_{n=2}^{k_1 + \cdots + k_p} I(t^{\sigma(n-1)}_{\beta(n-1)} < t^{\sigma(n)}_{\beta(n)}),$$

that is, we replace inequalities in the wide sense by strict inequalities. Note that we use the fact that $\sigma \in \Gamma(k_1, ..., k_p)$. After this replacement, we see that the Dirac measures involved in the definition of $\Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}$ give no contribution to the left-hand side of (3.g). Thus we can drop them and we obtain the right-hand side.

Then Proposition 2.3 implies that for $\sigma \in \Gamma(k_1, ..., k_p)$,

$$\left| E\left[ \int_{\Lambda_\sigma} \prod_{n=1}^{k_1+\cdots+k_p} A^{\sigma(n), y_{\sigma(n)}}_{\beta(n), \varepsilon_{\sigma(n)}}(dt^{\sigma(n)}_{\beta(n)}) \right] - \prod_{n=1}^{k_1+\cdots+k_p} G(y_{\sigma(n)} - y_{\sigma(n-1)}) \right|$$

$$\leqslant \varepsilon F_\sigma(y_1, ..., y_p), \tag{3.h}$$

for some function $F_\sigma \in L^1((\mathbb{R}^2)^p, \mathbb{R}_+)$. Lemma 3.2 now follows from (3.f), (3.g), (3.h), and the claim.

*Proof of the Claim.* We take $\sigma \in \Gamma^*(k_1, ..., k_p) - \Gamma(k_1, ..., k_p)$, and set

$$m = \sup\{n; \sigma(n) = \sigma(n-1)\}.$$

Note that $m$ is well-defined since $\sigma \notin \Gamma(k_1, ..., k_p)$. We also set $q = \sigma(m) = \sigma(m-1)$, and for $i = 1, ..., p$,

$$k'_i = |\{n; \sigma(n) = i, n \leqslant m-1\}| \leqslant k_i.$$

Let $\Lambda'_\sigma$ denote the set of all families $(t^i_j; 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant k'_i)$ such that for $n = 1, ..., m-1$,

$$t^{\sigma(n)}_{\beta(n)} \leqslant t^{\sigma(n+1)}_{\beta(n+1)} < \zeta.$$

Obviously, if $(t^i_j;\ 1 \leqslant i \leqslant p;\ 1 \leqslant j \leqslant k_i) \in \Lambda_\sigma$ we also have $(t^i_j;\ 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant k'_i) \in \Lambda'_\sigma$. For $s \geqslant 0$, set

$$U(s) = \int_{\Lambda'_\sigma \cap \{t^{\sigma(m-1)}_{\beta(m-1)} \leqslant s\}} \prod_{i=1}^{p} \left( A^{i, y_i}_{1, \varepsilon_i}(dt^i_1) \prod_{j=2}^{k'_i} (A^{i, y_i}_{j, \varepsilon_i}(dt^i_j) \right.$$

$$\left. + h^{i,j}_{\varepsilon_i}(y_i - B_{t^i_{j-1}})\, \delta_{(t^i_{j-1})}(dt^i_j)) \right).$$

The process $U(s)$ is continuous with finite variation on compact sets. Moreover the measure $U(ds)$ is supported on $\{s; |B_s - y_q| \leqslant \varepsilon_q\}$. This follows from our definition of $\Lambda'_\sigma$ and the fact that the additive functionals $A^{q, y_q}_{j, \varepsilon_q}$ are supported on this set.

Set $k = k_1 + \cdots + k_p$. We introduce the random variable

$$V = \int I(t^{\sigma(m+1)}_{\beta(m+1)} \leqslant \cdots \leqslant t^{\sigma(k)}_{\beta(k)} < \zeta) \prod_{n=m+1}^{k} A^{\sigma(n), y_{\sigma(n)}}_{\beta(n), \varepsilon_{\sigma(n)}}(dt^{\sigma(n)}_{\beta(n)})$$

$$(V = 1 \text{ if } m = k).$$

Then,

$$\int_{\Lambda_\sigma} \Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}(dt^i_j)$$

$$= \int_0^\zeta U(ds) \int_0^\zeta (A^{q, y_q}_{\beta(m), \varepsilon_q}(dt) + h^{q, \beta(m)}_{\varepsilon_q}(y_q - B_s)\, \delta_{(s)}(dt))\, V \circ \theta_t. \tag{3.i}$$

Formula (3.i) is a straightforward consequence of our definitions. The variables $(s, t)$ play the role of $(t^q_{\beta(m-1)}, t^q_{\beta(m)})$. Also note that in the definition of $V$ we have dropped all Dirac measures. As before, this is justified by the observation that for $m<n\leqslant k$, we have $\sigma(n-1)\neq\sigma(n)$, which implies that the additive functionals $A^{\sigma(n), y_{\sigma(n)}}_{\beta(n), \varepsilon_{\sigma(n)}}$, $A^{\sigma(n-1), y_{\sigma(n-1)}}_{\beta(n-1), \varepsilon_{\sigma(n-1)}}$ do not increase simultaneously.

We may now apply Lemma 2.2 to the right-hand side of (3.i). It follows that

$$\left|E\left[\int_{A_\sigma} \Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}(dt^i_j)\right]\right| \leqslant C\left(\log\frac{1}{\varepsilon}\right) E[U^*(\zeta)]\,\omega_\varepsilon(y_q, V), \tag{3.j}$$

with the notations of Proposition 2.1. It remains to bound $E[U^*(\zeta)]$ and $\omega_\varepsilon(y_q, V)$. First, the arguments which lead to (3.d) easily give the bound

$$E[U^*(\zeta)] \leqslant C\left(\log\frac{1}{\varepsilon}\right)^{m-2} \prod_{i\in I} G(y_i)^{1/|I|}, \tag{3.k}$$

where $I=\{i; k'_i\geqslant 1\}=\{i=\sigma(n)$ for some $n\leqslant m-1\}$. Second, we may apply Proposition 2.3 to $V$. It follows that

$$\omega_\varepsilon(y_q, V)\leqslant\varepsilon\, f(y_i-y_q, i\in J), \tag{3.l}$$

where $J=\{i;\ i\neq q$ and $i=\sigma(n)$ for some $n>m\}$ and $f$ is in $L^r((\mathbb{R}^2)^{|J|}, \mathbb{R}_+)$ for any $r\in[1, 2)$. Coming back to (3.j) we obtain

$$\left|E\left[\int_{A_\sigma} \Delta^{y_1, ..., y_p}_{\varepsilon_1, ..., \varepsilon_p}(dt^i_j)\right]\right| \leqslant \varepsilon\left(\log\frac{1}{\varepsilon}\right)^{m-1} F_\sigma(y_1, ..., y_p),$$

where

$$F_\sigma(y_1, ..., y_p)=C\left(\prod_{i\in I} G(y_i)^{1/|I|}\right) f(y_i, i\in J).$$

Noting that $I\cup J=\{1, ..., p\}$ we immediately see that $F_\sigma\in L^1((\mathbb{R}^2)^p, \mathbb{R}_+)$. This completes the proof of the claim. ∎

*Proof of Theorem* 3.1. We first establish the convergence in $L^2$-norm of $T^k_\varepsilon\varphi$ as $\varepsilon$ goes to zero. We have

$$T^k_\varepsilon\varphi=\int dy\,\varphi(y)\,X_\varepsilon(y),$$

where

$$X_\varepsilon(y)=\int_{D_k} A^{1,y}_\varepsilon(dt_1)\prod_{i=2}^{k}\left(A^{i,y}_\varepsilon(dt_i)+h^{A^i}_\varepsilon(y-B_{t_{i-1}})\,\delta_{(t_{i-1})}(dt_i)\right). \tag{3.m}$$

Hence

$$\begin{aligned}E[(T^k_\varepsilon\varphi - T^k_{\varepsilon'}\varphi)^2] &= \int dy\, dz\, \varphi(y)\, \varphi(z)(E[X_\varepsilon(y)\, X_\varepsilon(z)] \\ &\quad - 2E[X_\varepsilon(y) X_{\varepsilon'}(z)] + E[X_{\varepsilon'}(y) X_{\varepsilon'}(z)]).\end{aligned}$$

By a special case of Lemma 3.2 we obtain

$$E[(T^k_\varepsilon\varphi - T^k_{\varepsilon'}\varphi)^2] \leqslant C\, \|\varphi\|^2_\infty\, \varepsilon\left(\log\frac{1}{\varepsilon}\right)^{2k}, \tag{3.n}$$

provided $\varepsilon \in (0, \frac{1}{2})$ and $\varepsilon' \in [\varepsilon/2; \varepsilon]$. Modifying $C$ if necessary we may clearly assume that (3.n) holds for all $\varepsilon \in (0, \frac{1}{2})$, $\varepsilon' \in (0, \varepsilon]$. It follows that $(T^k_\varepsilon\varphi)$ is a Cauchy family in $L^2(\Omega, P)$. If $T^k\varphi$ denotes its limit we have

$$E[(T^k_\varepsilon\varphi - T^k\varphi)^2] \leqslant C\, \|\varphi\|^2_\infty\, \varepsilon\left(\log\frac{1}{\varepsilon}\right)^{2k}.$$

The same method gives the desired bound on the moment of order $2p$ of $T^k_\varepsilon\varphi - T^k\varphi$. In particular, we see that $T^k_\varepsilon\varphi$ converges to $T^k\varphi$ in $L^p(\Omega, P)$, for every $p < \infty$.

Suppose that we replace $A^1, ..., A^k$ by $k$ other additive functionals $\bar{A}^1, ..., \bar{A}^k \in \mathscr{A}_1$. Then, with obvious notations,

$$\begin{aligned}&\lim_{\varepsilon \to 0} E[(T^k_\varepsilon\varphi - \bar{T}^k_\varepsilon\varphi)^2] \\ &\quad = \lim_{\varepsilon \to 0} \int dy\, dz\, \varphi(y)\, \varphi(z)\, E[(X_\varepsilon(y) - \bar{X}_\varepsilon(y))(X_\varepsilon(z) - \bar{X}_\varepsilon(z))] = 0\end{aligned}$$

by Lemma 3.2 again. This shows that the definition of $T^k\varphi$ does not depend on the choice of the functionals $A^1, ..., A^k$.

Finally, the last assertion of Theorem 3.1 is a straightforward consequence of Lemma 3.2 and the fact that $T^k_\varepsilon\varphi$ converges to $T^k\varphi$ in $L^p$-norm for any $p < \infty$. ∎

(3.3) We will now describe another approximation of the fields $T^k\varphi$, which is essentially contained in Dynkin [7]. However, it seems worthwhile to point out that this approximation can be derived by the methods we have used to prove Theorem 3.1. Let $\mathscr{P}$ denote the set of bounded probability densities $q(y)$ on $\mathbb{R}^2$ with compact support. For $q \in \mathscr{P}$ and $\varepsilon > 0$ we set

$$q_\varepsilon(y) = \varepsilon^{-2} q(y/\varepsilon)$$

$$h^{(q)}_\varepsilon = E_0\left[\int_0^\zeta q_\varepsilon(B_s)\, ds\right] = E_y\left[\int_0^\zeta q_\varepsilon(B_s - y)\, ds\right] \tag{3.o}$$

for any $y \in \mathbb{R}^2$.

THEOREM 3.3. *Let $k \geq 1$ and $q_1, \ldots, q_k \in \mathscr{P}$. For $\varphi \in \mathscr{B}(\mathbb{R}^2, \mathbb{R}_+)$, set*

$$\tilde{T}_\varepsilon^k \varphi = \int dy\, \varphi(y) \int_{D_k} q_\varepsilon^1(W_{t_1} - y)\, dt_1 \prod_{i=2}^{k} (q_\varepsilon^i(B_{t_i} - B_{t_{i-1}})\, dt_i + h_\varepsilon^{(q^i)}\, \delta_{(t_{i-1})}(dt_i)).$$

*Then, for every integer $p \geq 1$, there exists a constant $C_p$, which does not depend on $\varphi$, $\varepsilon$, such that*

$$E[(T_\varepsilon^k \varphi - T^k \varphi)^{2p}] \leq C_p \, \|\varphi\|_\infty^{2p}\, \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{2pk},$$

*where $T^k\varphi$ is defined in Theorem* 3.1.

Dynkin [7] considers the case $q^1 = \cdots = q^k = q$, but with less restrictive assumptions on the function $q$. Instead of $\tilde{T}_\varepsilon^k \varphi$, he studies

$$\hat{T}_\varepsilon^k \varphi = \int_{D_k} \varphi(B_{t_1})\, dt_1 \prod_{i=2}^{k} (q_\varepsilon^i(B_{t_i} - B_{t_{i-1}})\, dt_i + h_\varepsilon^{(q^i)}\, \delta_{(t_{i-1})}(dt_i)).$$

This makes no real difference. Indeed, when $\varphi$ is Lipschitz it is easily seen that

$$E[|\tilde{T}_\varepsilon^k \varphi - \hat{T}_\varepsilon^k \varphi|] \leq C\varepsilon \left(\log \frac{1}{\varepsilon}\right)^k,$$

and then one can use some density argument to deal with bounded measurable functions.

The main advantage of the approximation given by Theorem 3.3 is that the "renormalization terms" $h_\varepsilon^{(q)}$ are now deterministic. This comes from the fact that the right-hand side of (3.o) does not depend on $y$. One should compare this with the situation of Theorem 3.1, where

$$h_\varepsilon^A(y) = E_y[A_\varepsilon^0(\zeta)]$$

obviously depends on $y$. In the special case when $q^1 = \cdots = q^k = q$, we have

$$\tilde{T}_\varepsilon^k \varphi = \sum_{n=1}^{k} \binom{k-1}{n-1} (h_\varepsilon^{(q)})^{k-n} \int dy\, \varphi(y) \times \int_{D_n} q_\varepsilon(W_{t_1} - y)\, dt_1 \prod_{i=2}^{n} q_\varepsilon(B_{t_i} - B_{t_{i-1}})\, dt_i.$$

The proof of Theorem 3.3 follows the general outline of that of Theorem 3.1. One writes

$$\tilde{T}_\varepsilon^k \varphi = \int dy\, \varphi(y)\, \tilde{X}_\varepsilon(y)$$

and the convergence of $\tilde{T}_\varepsilon^k\varphi$ as $\varepsilon \to 0$ follows by studying the asymptotics of

$$E[\tilde{X}_{\varepsilon_1}(y_1)\cdots\tilde{X}_{\varepsilon_p}(y_p)],$$

as in Lemma 3.2. It is necessary to obtain a version of Proposition 2.3 adapted to this new setting, but this causes no problem. In order to identify the limit with $T^k\varphi$, it suffices to prove that

$$\lim_{\varepsilon\to 0} E[X_\varepsilon(y)\,X_\varepsilon(z)] = \lim_{\varepsilon\to 0} E[X_\varepsilon(y)\,\tilde{X}_\varepsilon(z)] = \lim_{\varepsilon\to 0} E[\tilde{X}_\varepsilon(y)\,\tilde{X}_\varepsilon(z)].$$

Since the proof involves no new argument, we will leave details to the reader. Moreover, in the next subsection, we will describe in great detail another method of identifying the fields $T^k$ with those introduced by Dynkin [7].

(3.4) We will now apply Theorem 3.1 to the problem of renormalizing the powers of the occupation field of $B$. Let us first give some motivation for this problem, essentially taken from [6]. The occupation field of the process $B$ killed at time $\zeta$ is defined by

$$\tau^1\varphi = \int_0^\zeta \varphi(B_s)\,ds \qquad (\varphi \in \mathscr{B}(\mathbb{R}^2, \mathbb{R})).$$

A formal density for the field $\tau^1$ is given by

$$\int_0^\zeta \delta_{(z)}(B_s)\,ds \qquad (z \in \mathbb{R}^2),$$

which can be approximated by $A_\varepsilon^y(\zeta)$, for any $A \in \mathscr{A}_1$. In fact, as a special case of Theorem 3.1, we know that the fields

$$\varphi \to \tau_\varepsilon^1\varphi = \int_0^\zeta dy\,\varphi(y)\,A_\varepsilon^y(\zeta)$$

converge to $\tau^1$ as $\varepsilon$ tends to 0.

Let $n \geqslant 2$. If we now look for a definition of the $n$th-power of $\tau^1$, a natural idea is to study the asymptotics of

$$\tau_\varepsilon^n\varphi = \int_0^\zeta dy\,\varphi(y)(A_\varepsilon^y(\zeta))^n = n!\int dy\,\varphi(y)\int_{D_n} A_\varepsilon^y(dt_1)\cdots A_\varepsilon^y(dt_n).$$

The family $\tau_\varepsilon^n\varphi$ does not converge as $\varepsilon$ tends to 0. However, we will obtain (Theorem 3.4 below) an asymptotic decomposition of $\tau_\varepsilon^n\varphi$ which involves the variables $T^n\varphi$ defined in Theorem 3.1. In particular the "finite part" of this decomposition can be written in the form

$$n!\, T^n\varphi + \sum_{k=1}^{n-1} \alpha_k T^k\varphi,$$

where the coefficients $\alpha_k$ depend on the choice of $A$. Thus, in some sense, the field $T^n$ can be called the renormalized $n$th-power of the occupation field.

We need a few additional notations before stating Theorem 3.4. Let $A^1, ..., A^n \in \mathscr{A}$ and let $\nu^1, ..., \nu^n$ denote the associated measures on $\mathbb{R}^2$. For every subset $M$ of $\{1, ..., n\}$ of the type $M = \{i, i+1, ..., j\}$ we define a new measure $\nu^M$ as follows. We fix a parameter $\beta \in \mathbb{R}$ and, for $y \in \mathbb{R}^2$, we set

$$F(y) = \frac{1}{\pi} \log \frac{1}{|y|} + \frac{1}{\pi}\left(\frac{\log 2 - \log \lambda}{2} - \gamma - \beta\right)$$

(compare with (2.b)). Then $\nu^M$ is defined by induction by

$$\begin{aligned} \nu^M &= \nu^i && \text{if} \quad M = \{i\} \\ \nu^M &= (F * \nu^{M - \{i\}})\nu^i && \text{if} \quad M = \{i, i+1, ..., j\},\ i<j. \end{aligned}$$

We observe that, for every subinterval $M$ of $\{1, ..., n\}$, $\nu^M$ is supported on $D$ and the $\lambda$-potential of $|\nu^M|$ is bounded. Thus the additive functional $A^M$ associated with $\nu^M$ belongs to $\mathscr{A}$.

For $1 \leqslant p \leqslant n$, let $\mathscr{B}_{n,p}$ denote the set of all ordered partitions of $\{1, ..., n\}$ with $p$ elements. In other words, $\mathscr{B}_{n,p}$ consists of all $p$-tuples $(M_1, ..., M_p)$, where $M_1 = \{1, ..., i_1\}$, $M_2 = \{i_1 + 1, ..., i_2\}$, $M_p = \{i_{p-1}, ..., n\}$ for some $1 \leqslant i_1 < i_2 < \cdots < i_{p-1} < n$.

With these definitions at hand we can now state our main result. Recall that $\bar{\nu} = \nu(\mathbb{R}^2)$ denotes the total mass of the (signed) measure $\nu$.

THEOREM 3.4. *Let $A^1, ..., A^N \in \mathscr{A}$ and let $\nu^1, ..., \nu^n$ denote the corresponding measures on $\mathbb{R}^2$. Then, for every $\varphi \in \mathscr{B}(\mathbb{R}^2, \mathbb{R}_+)$,*

$$\int dy\, \varphi(y) \int_{D_n} A_\varepsilon^{1,y}(dt_1) \cdots A_\varepsilon^{n,y}(dt_n)$$
$$= \sum_{k=1}^{n} P_{n,k}\left(\frac{1}{\pi}\left(\log\frac{1}{\varepsilon} + \beta\right)\right) T^k\varphi + R_n(\varepsilon, \varphi),$$

*where the polynomials $P_{n,k}$ are defined by*

$$P_{n,k}(X) = \sum_{p=k}^{n} \binom{p-1}{k-1} \left( \sum_{(M_1, ..., M_p) \in \mathscr{B}_{n,p}} \prod_{i=1}^{p} \overline{\nu^{M_i}} \right) X^{p-k}$$

*and the rest* $R_n(\varepsilon, \varphi)$ *satisfies, for every* $p \geqslant 1$,

$$E[R_n(\varepsilon, \varphi)^{2p}] \leqslant C_p \, \|\varphi\|_\infty^{2p} \, \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{2np}$$

*for some constant* $C_p$ *independent of* $\varphi$ *and* $\varepsilon \in (0, \frac{1}{2})$.

*Proof.* Theorem 3.4 is a direct consequence of Theorem 3.1. Indeed, purely algebraic calculations show that the asymptotic behaviour of the functionals

$$\int dy \, \varphi(y) \int_{D_n} A_\varepsilon^{1, y}(dt_1) \cdots A_\varepsilon^{n, y}(dt_n)$$

can be reduced to that of functionals of the type investigated in Theorem 3.1. The idea is to write, for $i = 2, ..., p$,

$$A_\varepsilon^{i, y}(dt_i) = (A_\varepsilon^{i, y}(dt_i) + h_\varepsilon^{A^i}(y - B_{t_{i-1}}) \, \delta_{(t_{i-1})}(dt_i)) - h_\varepsilon^{A^i}(y - B_{t_{i-1}}) \, \delta_{(t_{i-1})}(dt_i)$$

and then to expand the product $A_\varepsilon^{1, y}(dt_1) \cdots A_\varepsilon^{n, y}(dt_n)$. We make this more explicit in the following lemma.

LEMMA 3.5. *For every* $y \in \mathbb{R}^2$ *and* $\varepsilon \in (0, \frac{1}{2})$, *the following identity of measures on* $D_n$ *holds*:

$$\begin{aligned}
&A_\varepsilon^{1, y}(dt_1) \cdots A_\varepsilon^{n, y}(dt_n) \\
&\quad = \sum_{p=1}^{n} \sum_{(M_1, ..., M_p) \in \mathscr{B}_{n,p}} \sum_{\{1\} \subset I \subset \{1, ..., p\}} \left( \prod_{i \in \{1, ..., p\} - I} \left( \frac{1}{\pi} \left( \log \frac{1}{\varepsilon} + \beta \right) \overline{v^{M_i}} \right) \right) \\
&\quad\quad \times \left( A_\varepsilon^{M_1, y}(dt_1) \prod_{i \in I - \{1\}} (A_\varepsilon^{M_i, y}(dt_{m_i}) \right. \\
&\quad\quad \left. + h_\varepsilon^{M_i}(y - B_{t_{m_i - 1}}) \, \delta_{(t_{m_i - 1})}(dt_{m_i})) \right) \\
&\quad\quad \times \left( \prod_{j \notin \{m_i, i \in I\}} \delta_{(t_{j-1})}(dt_j) \right) + \chi_n(\varepsilon, y)(dt_1 \cdots dt_n),
\end{aligned}$$

*where* $m_i$ *denotes* $\inf\{j; j \in M_i\}$ *and*

$$E\left[ \left( \int dy \, |\chi_n(\varepsilon, y)| \, (D_n) \right)^p \right] \leqslant C_1 \varepsilon \left( \log \frac{1}{\varepsilon} \right)^{np}$$

*for every* $p \geqslant 1$, *for some constant* $C_p$.

Theorem 3.4 follows from Lemma 3.5, Theorem 3.1 (see also remark (ii) after this theorem), and the trivial observation that

$$\int_{D_n} A_\varepsilon^{M_1, y}(dt_1) \prod_{i \in I - \{1\}} (A_\varepsilon^{M_i, y}(dt_{m_i}) + h_\varepsilon^{M_i}(y - B_{t_{m_i - 1}})\, \delta_{(t_{m_i - 1})}(dt_{m_i}))$$

$$\times \prod_{j \notin \{m_i, i \in I\}} \delta_{(t_{j-1})}(dt_j)$$

$$= \int_{D_k} \tilde{A}_\varepsilon^{1, y}(dt_1) \prod_{j=2}^{k} (\tilde{A}_\varepsilon^{j, y}(dt_j) + h_\varepsilon^{\tilde{A}_j, y}(y - B_{t_{j-1}})\, \delta_{(t_{j-1})}(dt_j)),$$

where $k = |I|$ and $\tilde{A}^j = A^{M_{i_j}}$, if $I = \{i_1 < i_2 < \cdots < i_k\}$.

*Proof of Lemma* 3.5. We proceed by induction on $n$. The assertion of the lemma is trivial for $n = 1$. Suppose it holds at the order $n$. We then have

$$A_\varepsilon^{n+1, y}(dt_{n+1}) = (A_\varepsilon^{n+1, y}(dt_{n+1}) + h_\varepsilon^{A^{n+1}}(y - B_{t_n})\, \delta_{(t_n)}(dt_{n+1}))$$
$$- h_\varepsilon^{A^{n+1}}(y - B_{t_n})\, \delta_{(t_n)}(dt_{n+1}).$$

By (2.b) and the definition of $h_\varepsilon^A$, we have, for $|z| \leqslant \varepsilon$,

$$-h_\varepsilon^{A^{n+1}}(z) = \frac{1}{\pi}\left(\log\frac{1}{\varepsilon} + \beta\right)\overline{v^{n+1}} + F * v^{n+1}\left(-\frac{z}{\varepsilon}\right) + \eta(\varepsilon, z),$$

where $|\eta(\varepsilon, z)| \leqslant C\varepsilon^2 \log(1/\varepsilon)$. Note that

$$F * v^{n+1}\left(\frac{B_{t_n} - y}{\varepsilon}\right) A_\varepsilon^{n, y}(dt_n) = A_\varepsilon^{\{n, n+1\}}(dt_n),$$

by definition of $A^{\{n, n+1\}}$. If follows that

$$A_\varepsilon^{n, y}(dt_n)\, A_\varepsilon^{n+1, y}(dt_{n+1})$$

$$= A_\varepsilon^{n, y}(dt_n)(A_\varepsilon^{n+1, y}(dt_{n+1}) + h_\varepsilon^{A^{n+1}}(y - B_{t_n})\, \delta_{(t_n)}(dt_{n+1}))$$

$$+ \left(\frac{1}{\pi}\left(\log\frac{1}{\varepsilon} + \beta\right)\overline{v^{n+1}} A_\varepsilon^{n, y}(dt_n) + A_\varepsilon^{\{n, n+1\}, y}(dt_n)\right) \delta_{(t_n)}(dt_{n+1})$$

$$+ \eta(\varepsilon, y - B_{t_n})\, A_\varepsilon^{n, y}(dt_n)\, \delta_{(t_n)}(dt_{n+1}). \tag{3.p}$$

The desired result at the order $n + 1$ follows from (3.p) by applying the induction hypothesis both to $A_\varepsilon^{1, y}(dt_1) \cdots A_\varepsilon^{n, y}(dt_n)$ and to $A_\varepsilon^{1, y}(dt_1) \cdots A_\varepsilon^{\{n, n+1\}, y}(dt_n)$. ∎

*Remarks.* (i) Suppose that $A^1 = A^2 = \cdots = A^n = l$, the local time of $B$ on the unit circle, normalized so that the associated measure has total mass 1. Choose $\beta$ such that $F(y) = (1/\pi)\log(1/|y|)$. Then $v^M = 0$ as soon as $|M| \geqslant 2$. It follows that, in this case,

$$P_{n,k}(X) = \binom{n-1}{k-1} X^{n-k}.$$

Of course we could have obtained this more directly: see remark (i) after Theorem 3.1.

(ii) Suppose that $A^1(dt) = \cdots = A^n(dt) = q(B_t)\,dt$, for some bounded measurable function $q$ supported on the closed unit disk, and take $\lambda = 1$, $\beta = 0$. Then Theorem 3.4 reduces to formula (1.28) of Dynkin [7]. In particular, the polynomials $M_{nk}$ defined by formula (1.29) of [7] coincide with the $P_{n,k}$ of Theorem 3.4. In consequence, the fields $T^k$ coincide with those introduced by Dynkin. To be specific, in Dynkin's notations, $T^k\varphi = \tau_k(\lambda, \zeta)$, where $\lambda(dx) = \varphi(x)\,dx$.

## 4. The Main Estimate for the Wiener Sausage

(4.1) Throughout this section we consider a compact subset $K$ of $\mathbb{R}^2$. We assume that $K$ has positive logarithmic capacity and is contained in the closed unit disk. For $\varepsilon > 0$, $S_\varepsilon^K(0, t)$ denotes the Wiener sausage associated with $\varepsilon K$ on the time interval $[0; t]$:

$$S_\varepsilon^K(0, t) = \bigcup_{0 \leqslant s \leqslant t} (B_s + \varepsilon K).$$

We also consider the random field $\varphi \to S_\varepsilon^K \varphi$ defined for $\varphi \in \mathscr{B}(\mathbb{R}^2, \mathbb{R}_+)$ by

$$S_\varepsilon^K \varphi = \int dy\, \varphi(y)\, I(y \in S_\varepsilon^K(0, \zeta)).$$

In particular, $S_\varepsilon^K 1$ is simply the area of the sausage $S_\varepsilon^K(0, \zeta)$. The goal of this section is to prove that the field $S_\varepsilon^K \varphi$ can be expressed in terms of the fields $T^k\varphi$ of the previous section, up to an error term smaller than $|\log \varepsilon|^{-n}$, where $n$ is a positive integer which can be taken arbitrarily large. As in Section 2 we define

$$h_\varepsilon^K = -\operatorname{cap}(\varepsilon K)^{-1},$$

where cap($\cdot$) denotes the capacity associated with Brownian motion killed at time $\zeta$. Recall that

$$h_\varepsilon^K = -\frac{1}{\pi}\left(\log\frac{1}{\varepsilon}+\frac{\log 2-\log\lambda}{2}-\gamma+R(K)\right)+O\left(\varepsilon^2\log\frac{1}{\varepsilon}\right),$$

where the constant $R(K)$ is defined by (2.g).

THEOREM 4.1. *Let $n\geqslant 1$ and $\varphi\in\mathscr{B}(\mathbb{R}^2, \mathbb{R}_+)$. Then,*

$$S_\varepsilon^K\varphi = -\sum_{k=1}^{n} (h_\varepsilon^K)^{-k}\, T^k\varphi + R_n(\varepsilon, \varphi),$$

*where the rest $R_n(\varepsilon,\varphi)$ satisfies*

$$E[R_n(\varepsilon,\varphi)^2]\leqslant C(h_\varepsilon^K)^{-2(n+1)}$$

*for some constant $C$ and for $\varepsilon\in(0;1)$.*

*Remark.* It would be possible to obtain $L^p$-bounds for $R_n(\varepsilon,\varphi)$. Since the proof involves a lot of technical details we will leave this refinement to the reader.

With each additive functional $A\in\mathscr{A}_1$ we can associate an approximation $T_\varepsilon^k$ of $T^k$, as in Theorem 3.1, by setting

$$\begin{aligned} T_\varepsilon^k\varphi &= \int dy\,\varphi(y)\int_{D_k} A_\varepsilon^y(dt_1)\prod_{i=2}^{k}\left(A_\varepsilon^y(dt_i)+h_\varepsilon^A(y-B_{t_{i-1}})\,\delta_{(t_{i-1})}(dt_i)\right)\\ &= \int dy\,\varphi(y)\,X_\varepsilon^{(k)}(y). \end{aligned}$$

From now on we fix $A\in\mathscr{A}_1$. In view of the bounds of Theorem 3.1 it suffices to prove the statement of Theorem 4.1 with $T^k\varphi$ replaced by $T_\varepsilon^k\varphi$, for $k=1, ..., n$.

For $\varepsilon>0$, $y\in\mathbb{R}^2$, set

$$Y_\varepsilon(y)=I(y\in S_\varepsilon^K(0,\zeta)).$$

Then,

$$\begin{aligned} &E\left[\left(S_\varepsilon^K\varphi+\sum_{k=1}^{n}(h_\varepsilon^K)^{-k}\,T_\varepsilon^k\varphi\right)^2\right]\\ &\quad=\int dy\,dz\,\varphi(y)\,\varphi(z)\,E\left[\left(Y_\varepsilon(y)+\sum_{k=1}^{n}(h_\varepsilon^K)^{-k}X_\varepsilon^{(k)}(y)\right)\right.\\ &\qquad\left.\times\left(Y_\varepsilon(z)+\sum_{k=1}^{n}(h_\varepsilon^K)^{-k}X_\varepsilon^{(k)}(z)\right)\right]. \end{aligned}$$

Thus, the proof of Theorem 4.1 reduces to that of the following lemma.

LEMMA 4.2. *There exists a function* $F \in L^1((\mathbb{R}^2)^2, \mathbb{R}_+)$ *such that, for* $\varepsilon \in (0, \frac{1}{2})$ *and* $y, z$ *two distinct points of* $\mathbb{R}^2 - \{0\}$,

$$\left| E\left[\left(Y_\varepsilon(y) + \sum_{k=1}^{n} (h_\varepsilon^K)^{-k} X_\varepsilon^{(k)}(y)\right)\left(Y_\varepsilon(z) + \sum_{k=1}^{n} (h_\varepsilon^K)^{-k} X_\varepsilon^{(k)}(z)\right)\right]\right|$$

$$\leqslant (h_\varepsilon^K)^{-2(n+1)} F(y, z).$$

(4.2) The proof of Lemma 4.2 requires asymptotics as $\varepsilon \to 0$ for the three following quantities:

(i) $E[X_\varepsilon^{(k)}(y)\, X_\varepsilon^{(l)}(z)]$ $(1 \leqslant k, l \leqslant n)$,

(ii) $E[Y_\varepsilon(y)\, X_\varepsilon^{(k)}(z)]$ $(1 \leqslant k \leqslant n)$,

(iii) $E[Y_\varepsilon(y)\, Y_\varepsilon(z)]$.

Case (i) follows from Lemma 3.2. Cases (ii) and (iii) are treated in the next two lemmas.

LEMMA 4.3. *For every* $p \geqslant 2$, *there exists a function* $F_p \in L^1((\mathbb{R}^2)^2, \mathbb{R}_+)$ *such that*, for $\varepsilon \in (0, \frac{1}{2})$ *and* $y, z \in \mathbb{R}^2 - \{0\}$, $y \neq z$,

$$\left| E[Y_\varepsilon(y)\, Y_\varepsilon(z)] - \sum_{k=2}^{p} (h_\varepsilon^K)^{-k} (G(y) + G(z))\, G(y-z)^{k-1} \right|$$

$$\leqslant \left(\log \frac{1}{\varepsilon}\right)^{-p-1} F_p(y, z).$$

*Proof.* The proof of Lemma 4.3 is not difficult and uses standard arguments. We shall give details for the cases $p = 2, 3$. It will then be clear that the proof can be continued by induction on $p$. It suffices to treat the case when $|y| \geqslant 4\varepsilon$, $|y - z| \geqslant 4\varepsilon$. Then,

$$E[Y_\varepsilon(y)\, Y_\varepsilon(z)] = P[T_\varepsilon(y) \leqslant T_\varepsilon(z) < \zeta] + P[T_\varepsilon(z) \leqslant T_\varepsilon(y) < \zeta],$$

where

$$T_\varepsilon(y) = T_{\varepsilon K}(y) = \inf\{t \geqslant 0;\, B_t \in y - \varepsilon K\}.$$

By (2.h),

$$|h_\varepsilon^K P[T_\varepsilon(y) < \zeta] + G(y)| \leqslant 2\varepsilon \rho(|y|) \tag{4.a}$$

and, for any $w \in y - \varepsilon K$,

$$|h_\varepsilon^K P_w[T_\varepsilon(z) < \zeta] + G(z-y)| \leqslant 2\varepsilon\rho(|z-y|). \tag{4.b}$$

Then,

$$\begin{aligned} &P[T_\varepsilon(y) \leqslant T_\varepsilon(z) < \zeta] \\ &\quad = P[T_\varepsilon(y) \leqslant T_\varepsilon'(z) < \zeta] - P[T_\varepsilon(z) \leqslant T_\varepsilon(y) \leqslant T_\varepsilon'(z) < \zeta], \end{aligned} \tag{4.c}$$

where $T_\varepsilon'(z) = \inf\{t > T_\varepsilon(y);\ z \in y - \varepsilon K\}$. Applying the strong Markov property at time $T_\varepsilon(y)$ leads to

$$\begin{aligned} P[T_\varepsilon(y) \leqslant T_\varepsilon'(z) < \zeta] &= E[(T_\varepsilon(y) < \zeta)\, P_{B_{T_\varepsilon(y)}}[T_\varepsilon(z) < \zeta]] \\ &= (h_\varepsilon^K)^{-2} G(y)\, G(z-y) + R(\varepsilon, y, z), \end{aligned} \tag{4.d}$$

where, according to (4.a), (4.b), and Lemma 2.1(i),

$$\begin{aligned} |R(\varepsilon, y, z)| &\leqslant 2\varepsilon((-h_\varepsilon^K)^{-1}\rho(|z-y|)\, P[T_\varepsilon(y) < \zeta] + (h_\varepsilon^K)^{-2}\rho(|y|)\, G(z-y)) \\ &\leqslant C\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{-2}\left(G\left(\frac{y}{2}\right)\rho(|z-y|) + \rho(|y|)\, G(z-y)\right). \end{aligned}$$

On the other hand, by Lemma 2.1(i) and two successive applications of the strong Markov property,

$$P[T_\varepsilon(z) \leqslant T_\varepsilon(y) \leqslant T_\varepsilon'(z) < \zeta] \leqslant C\left(\log\frac{1}{\varepsilon}\right)^3 G\left(\frac{z}{2}\right) G\left(\frac{z-y}{2}\right)^2. \tag{4.e}$$

The case $p = 2$ follows from (4.c), (4.d), and (4.e).

Now consider the case $p = 3$. Instead of (4.e) we write

$$\begin{aligned} &P[T_\varepsilon(z) \leqslant T_\varepsilon(y) \leqslant T_\varepsilon'(z) < \zeta] \\ &\quad = P[T_\varepsilon(z) \leqslant T_\varepsilon'(y) \leqslant T_\varepsilon''(z) < \zeta] \\ &\qquad - P[T_\varepsilon(y) \leqslant T_\varepsilon(z) \leqslant T_\varepsilon'(y) \leqslant T_\varepsilon''(z) < \zeta], \end{aligned}$$

where

$$\begin{aligned} T_\varepsilon'(y) &= \inf\{t > T_\varepsilon(z) \colon B_t \in y - \varepsilon K\}, \\ T_\varepsilon''(z) &= \inf\{t > T_\varepsilon'(y);\ B_t \in z - \varepsilon K\}. \end{aligned}$$

It follows from (4.a) and (4.b) that

$$P[T_\varepsilon(z) \leqslant T_\varepsilon'(y) \leqslant T_\varepsilon''(z) < \zeta] = -(h_\varepsilon^K)^{-3} G(z)\, G(z-y)^2 + R'(\varepsilon, y, z), \tag{4.f}$$

where

$$|R'(\varepsilon, y, z)| \leqslant C\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{-3}$$
$$\times\left(\rho(|y|)\,G(z-y)^2 + G\left(\frac{y}{2}\right)G\left(\frac{z-y}{2}\right)\rho(|z-y|)\right).$$

On the other hand, as in (4.e) above

$$P[T_\varepsilon(y) \leqslant T_\varepsilon(z) \leqslant T'_\varepsilon(y) \leqslant T''_\varepsilon(z) < \zeta]$$
$$\leqslant C\left(\log\frac{1}{\varepsilon}\right)^{-4} G\left(\frac{y}{2}\right)G\left(\frac{z-y}{2}\right)^3. \tag{4.g}$$

The case $p=3$ follows from (4.d), (4.f), and (4.g).

It should now be clear to the reader that the general case can easily be treated by induction. The function $F_p$ may be defined by

$$F_p(y, z) = C\left[(\rho(|y|) + \rho(|z|))\sum_{i=1}^{p-1} G\left(\frac{z-y}{2}\right)^i\right.$$
$$+\left(G\left(\frac{y}{2}\right) + G\left(\frac{z}{2}\right)\right)$$
$$\left.\times\left(\rho(|z-y|)\sum_{i=1}^{p-2} G\left(\frac{z-y}{2}\right)^i + G\left(\frac{z-y}{2}\right)^p\right)\right],$$

which is clearly integrable. ∎

LEMMA 4.4. *Let $k \geqslant 1$. There exists a function $F'_k \in L^1((\mathbb{R}^2)^2, \mathbb{R}_+)$ such that, for $\varepsilon \in (0, \frac{1}{2})$ and $y, z \in \mathbb{R}^2 - \{0\}$, $y \neq z$,*

— *if $k \geqslant 2$,*

$$|E[Y_\varepsilon(y)\,X_\varepsilon^{(k)}(z)] + (h_\varepsilon^K)^{-k+1}G(z)\,G(z-y)^{2k-2}$$
$$+ (h_\varepsilon^K)^{-k}(G(y) + G(z))\,G(z-y)^{2k-1} + (h_\varepsilon^K)^{-k-1}G(y)\,G(z-y)^{2k}|$$
$$\leqslant \varepsilon\left(\log\frac{1}{\varepsilon}\right)^k F'_k(y, z),$$

— *if $k=1$,*

$$|E[Y_\varepsilon(y)\,X_\varepsilon^{(1)}(z)]$$
$$+ (h_\varepsilon^K)^{-1}(G(y) + G(z))\,G(z-y) + (h_\varepsilon^K)^{-2}G(y)\,G(z-y)^2|$$
$$\leqslant \varepsilon\left(\log\frac{1}{\varepsilon}\right) F'_1(y, z).$$

*Proof.* We only consider the case $k \geqslant 2$. The case $k=1$ is similar and easier. We first note that

$$\begin{aligned}
&|E[Y_\varepsilon(y)\, X_\varepsilon^{(k)}(z)]| \\
&\quad \leqslant P[y \in S_\varepsilon^D(0,\zeta),\, z \in S_\varepsilon^D(0,\zeta)]^{1/2} E[X_\varepsilon^{(k)}(z)^2]^{1/2} \\
&\quad \leqslant C\left(\log\frac{1}{\varepsilon}\right)^{k-1}\left(G\left(\frac{y}{2}\right)G\left(\frac{z-y}{2}\right)+G\left(\frac{z}{2}\right)G\left(\frac{y-z}{2}\right)\right)^{1/2},
\end{aligned}$$

where we use Lemma 2.1(ii) and the easy bound

$$E[X_\varepsilon^{(k)}(z)^2] \leqslant C\left(\log\frac{1}{\varepsilon}\right)^{2k}.$$

The same arguments as in the beginning of the proof of Lemma 3.2 now show that we may restrict our attention to the case $|y| \geqslant 4\varepsilon$, $|z| \geqslant 4\varepsilon$, $|z-y| \geqslant 4\varepsilon$. From now on, we make this assumption.

By definition, we have

$$\begin{aligned}
&E[Y_\varepsilon(y)\, X_\varepsilon^{(k)}(z)] \\
&\quad = E\left[I(y \in S_\varepsilon^K(0,\zeta)) \int_{D_k(\zeta)} A_\varepsilon^z(dt_1)\right. \\
&\qquad \left. \times \prod_{i=2}^{k} (A_\varepsilon^z(dt_i) + h_\varepsilon^K(z - B_{t_{i-1}})\, \delta_{(t_{i-1})}(dt_i))\right].
\end{aligned}$$

Let $0 \leqslant t_1 \leqslant \cdots \leqslant t_k$. On $\{\zeta > t_k\}$ we have

$$\{y \in S_\varepsilon^K(0,\zeta)\} = \bigcup_{i=0}^{k} \{y \in S_\varepsilon^K(t_i, t_{i+1})\},$$

with the convention that $t_0 = 0$, $t_{k+1} = \zeta$. For any family $(M_i, 0 \leqslant i \leqslant k)$ of subsets of $\mathbb{R}^2$, we have

$$I\left(y \in \bigcup_{i=0}^{k} M_i\right) = \sum_{L \in \mathscr{P}_k} (-1)^{|L|+1} I\left(y \in \bigcap_{i \in L} M_i\right),$$

where $\mathscr{P}_k$ denotes the set of nonempty subsets of $\{0, 1, ..., k\}$. Thus, taking $M_i = S_\varepsilon^K(t_i, t_{i+1})$,

$$E[Y_\varepsilon(y)\, X_\varepsilon^{(k)}(z)] = \sum_{L \in \mathscr{P}_k} (-1)^{|L|+1} \phi_L(\varepsilon, y, z), \tag{4.h}$$

where

$$\phi_L(\varepsilon, y, z) = E\left[\int_{D_k(\zeta)} \left(\prod_{i \in L} I(y \in S_\varepsilon^K(t_i, t_{i+1}))\right) A_\varepsilon^z(dt_1) \times \prod_{i=2}^{k} (A_\varepsilon^z(dt_i) + h_\varepsilon^K(z - B_{t_{i-1}})\, \delta_{(t_{i-1})}(dt_i))\right]. \tag{4.i}$$

*Claim.* Except when $\{1, ..., k-1\} \subset L$, there exists a function $F_L \in L^1((\mathbb{R}^2)^2, \mathbb{R}_+)$ such that, for $\varepsilon \in (0, \frac{1}{2})$ and $y, z \in \mathbb{R}^2 - \{0\}$, $y \neq z$,

$$|\phi_L(\varepsilon, y, z)| \leqslant \varepsilon \left(\log \frac{1}{\varepsilon}\right)^k F_L(y, z).$$

Assume that the claim is proved. In the right-hand side of (4.h), we need only consider the four cases $L = \{1, ..., k-1\}$, $L = \{1, ..., k\}$, $L = \{0, 1, ..., k-1\}$, and $L = \{0, 1, ..., k\}$. In each of these cases we can apply Proposition 2.4 to get the desired result. We first note that, when $\{1, ..., k-1\} \subset L$,

$$\phi_L(\varepsilon, y, z) = E\left[\int_{D_k(\zeta)} \prod_{i \in L} I(y \in S_\varepsilon^K(t_i, t_{i+1})) \prod_{i=1}^{k} A_\varepsilon^z(dt_i)\right].$$

In other words, we may drop the Dirac measures in the right-hand side of (4.i). Indeed, if $t_i = t_{i+1}$ belongs to the support of $A_\varepsilon^z(dt)$, we cannot have $y \in S_\varepsilon^K(t_i, t_{i+1})$, since $|y - z| \geqslant 4\varepsilon$. Then, for instance in the case $L = \{1, ..., k-1\}$, Proposition 2.4 implies that

$$|\phi_L(\varepsilon, y, z) - (-h_\varepsilon^K)^{-k+1} G(z)\, G(z-y)^{2k-2}| \leqslant \varepsilon\, |\log \varepsilon|^k F_L(y, z),$$

for some function $F_L \in L^1((\mathbb{R}^2)^2, \mathbb{R}_+)$. The three other possible cases can also be handled using Proposition 2.4. The bound of Lemma 4.4 then follows from (4.h).

*Proof of the Claim.* This is very similar to the proof of Lemma 3.2. We set

$$p = \sup\{i \in \{1, ..., k-1\};\, i \notin L\}.$$

This makes sense if $\{1, ..., k-1\}$ is not contained in $L$, which we now assume. Set $L' = L \cap \{0, 1, ..., p-1\}$ and $k' = k$ if $k \in L$, $k-1$ if $k \notin L$. Then,

$$\phi_L(\varepsilon, y, z) = E\left[\int_0^\zeta U(ds) \int_s^\zeta (A_\varepsilon^z(dt) + h_\varepsilon^K(z - B_s)\, \delta_{(s)}(dt))\, V \circ \theta_t\right], \tag{4.j}$$

where the process $U$ is defined by

$$U(s)=\int_{D_p(s)} \prod_{i\in L'} I(y\in S_\varepsilon^K(t_i, t_{i+1}))\, A_\varepsilon^z(dt_1)$$
$$\times \prod_{i=2}^{p} (A_\varepsilon^z(dt_i)+h_\varepsilon^K(z-B_{t_{i-1}})\,\delta_{(t_{i-1})}(dt_i))$$

and

$$V=\int_{D_{k-1-p}(\zeta)} \prod_{i=0}^{k'-1-p} I(y\in S_\varepsilon^K(t_i, t_{i+1})) \prod_{i=1}^{k-1-p} A_\varepsilon^z(dt_i).$$

Here we agree as usual that $t_0=0$, and, in the formula for $V$, we also make the convention $t_{k-p}=\zeta$. Formula (4.j) is a simple rewriting of (4.i), except for the fact that we have dropped the Dirac measures in the expression of $V$. This can be easily justified by the argument we have just used at the end of the proof of Lemma 4.4.

We can apply Lemma 2.2 to the right-hand side of (4.j):

$$|\phi_L(\varepsilon, y, z)|\leqslant C\left(\log\frac{1}{\varepsilon}\right) E[U^*(\zeta)]\,\omega_\varepsilon(z, V). \tag{4.k}$$

We first bound $E[U^*(\zeta)]$. If $L'\neq\varnothing$, the Hölder inequality leads to

$$E[U^*(\zeta)]\leqslant CP[y\in S_\varepsilon^D(0,\zeta), z\in S_\varepsilon^D(0,\zeta)]^{1/2} \sup_{x\in\mathbb{R}^2} E_x[A_\varepsilon^z(\zeta)^{2p}]^{1/2}$$
$$\leqslant C'\left(\log\frac{1}{\varepsilon}\right)^{p-1}\left((G(y)+G(z))\,G\left(\frac{y-z}{2}\right)\right)^{1/2}.$$

If $L'=\varnothing$, the same method gives

$$E[U^*(\zeta)]\leqslant C\left(\log\frac{1}{\varepsilon}\right)^{p-1/2} G\left(\frac{z}{2}\right)^{1/2}.$$

On the other hand, Proposition 2.4 implies

$$\omega_\varepsilon(z, V)\leqslant \varepsilon\, f(z-y),$$

for some function $f\in L^r(\mathbb{R}^2, \mathbb{R}_+)$, for any $r\in[1, 2)$. To be specific, we can apply Proposition 2.4 except in the case when $p=k-1$, $k'=k-1$, but in this case we have $V=1$ and thus $\omega_\varepsilon(z, V)=0$.

The claim follows from (4.k) and the above bounds. ∎

*Proof of Lemma* 4.2. A special case of Lemma 3.2 shows that, for $1 \leqslant k \leqslant l \leqslant n$,

— if $l-k \geqslant 2$,

$$|E[X_\varepsilon^{(k)}(y)\,X_\varepsilon^{(l)}(z)]| \leqslant \varepsilon\left(\log\frac{1}{\varepsilon}\right)^{k+l} F_{k,l}(y,z),$$

— if $l=k+1$,

$$|E[X_\varepsilon^{(k)}(y)\,X_\varepsilon^{(k+1)}(z)] - G(z)\,G(z-y)^{2k}| \leqslant \varepsilon\left(\log\frac{1}{\varepsilon}\right)^{2k+1} F_{k,l}(y,z),$$

— if $l=k$,

$$|E[X_\varepsilon^{(k)}(y)\,X_\varepsilon^{(k)}(z)] - (G(y)+G(z))\,G(z-y)^{2k-1}| \leqslant \varepsilon\left(\log\frac{1}{\varepsilon}\right)^{2k} F_{k,l}(y,z)$$

for some function $F_{k,l} \in L^1((\mathbb{R}^2)^2, \mathbb{R}_+)$. Lemma 4.2 is a direct consequence of these bounds and those of Lemma 4.3 (applied with $p=2n+1$) and Lemma 4.4. ∎

## 5. Asymptotic Expansions for the Area of the Wiener Sausage

(5.1) The previous three sections were devoted to some results concerning Brownian motion stopped at an independent exponential time $\zeta$. We now propose to investigate similar results for Brownian motion stopped at a constant time $t$. In particular, we shall introduce random variables $T^k(t)$ related to $k$-multiple intersections of the Brownian path on the time interval $[0; t]$. This will be done in such a way that $T^k(t)$ coincides with $T^k 1$, conditionally on $\{\zeta = t\}$. The conditioning requires some rigorous justification since $\{\zeta = t\}$ is a set of zero probability. Once the random variables $T^k(t)$ have been rigorously defined, we can prove asymptotic expansions for the area of the sausage $S_\varepsilon^K(0, t)$, which will involve these random variables.

THEOREM 5.1. *There exists a unique family* $(T^k(t);\ t>0,\ k=1,2,\ldots)$ *of random variables belonging to* $\mathscr{L}(\Omega_0, W)$ *that satisfies the following properties*:

(i) *$P(d\omega\, dt)$ a.s.,*

$$T^k 1(\omega, t) = T^k(t)(\omega).$$

(ii) *For any $t > 0$, $r > 0$, $W(d\omega)$ a.s.,*

$$T^k(rt)(\omega) = r \sum_{j=1}^{k} \left(\frac{\log r}{2\pi}\right)^{k-j} \binom{k-1}{j-1} T^j(t)(\omega_r),$$

*where $\omega_r \in \Omega_0$ is defined by $\omega_r(s) = r^{-1/2}\omega(rs)$.*

*Remarks.* Formula (i) cannot be used to define $T^k(t)$. Indeed, since $T^k 1(\omega, t)$ is only defined $P(d\omega\, dt)$ a.s., for fixed $t$, the mapping $\omega \to T^k 1(\omega, t)$ does not define a random variable. Formula (ii) gives a scaling property which can be interpreted as follows. Let $\tilde{T}^k_\varepsilon$ be defined as in Theorem 3.3, but with $q^1 = \cdots = q^k = q$. For $\varphi = 1$,

$$\tilde{T}^k_\varepsilon 1 = \int dy \int_{D_k} q_\varepsilon(B_{t_1} - y)\, dt_1 \times \prod_{i=2}^{k} \left(q_\varepsilon(B_{t_i} - B_{t_{i-1}})\, dt_i + h^{(q)}_\varepsilon\, \delta_{(t_{i-1})}(dt_i)\right).$$

Note that

$$h^{(q)}_{r^{-1/2}\varepsilon} = h^{(q)}_\varepsilon - \frac{\log r}{2\pi} + O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right).$$

A scaling transformation leads to

$$\tilde{T}^k_\varepsilon 1(\omega, rt) = r \sum_{j=1}^{k} \binom{k-1}{j-1} \left(\frac{\log r}{2\pi}\right)^{k-j} \tilde{T}^j_{r^{-1/2}\varepsilon} 1(\omega_r, t) + O(\varepsilon).$$

Hence, passing to the limit as $\varepsilon \to 0$,

$$T^k 1(\omega, rt) = r \sum_{j=1}^{k} \binom{k-1}{j-1} \left(\frac{\log r}{2\pi}\right)^{k-j} T^j 1(\omega_r, t).$$

Note that, in contrast with (ii), the latter identity only holds $P(d\omega\, dt)$ a.s.

THEOREM 5.2. *Let $n \geq 1$ and let $K$ be a compact subset of $\mathbb{R}^2$. Assume that $K$ has positive capacity. Then, for any $t > 0$,*

$$m(S^K_\varepsilon(0, t)) = -\sum_{k=1}^{n} (h^K_\varepsilon)^{-k} T^k(t) + R_n(\varepsilon, t),$$

*where the rest* $R_n(\varepsilon, t)$ *satisfies*

$$\lim_{\varepsilon \to 0} (h_\varepsilon^K)^n R_n(\varepsilon, t) = 0,$$

*in* $L^2$*-norm, and almost surely if* $K$ *is star-shaped, i.e.,* $\varepsilon K \subset K$ *for every* $\varepsilon \in [0; 1]$.

(5.2) *Proof of Theorems* 5.1 *and* 5.2. Let $K$ be a star-shaped compact subset of $\mathbb{R}^2$ with positive capacity. Set $N = 4n$. By Theorem 4.1,

$$E\left[\left(S_\varepsilon^K 1 + \sum_{k=1}^{N} (h_\varepsilon^K)^{-k} T_\varepsilon^k 1\right)^2\right] \leqslant C_N (h_\varepsilon^K)^{-2(N+1)}, \tag{5.a}$$

or, equivalently,

$$\int_0^\infty dt\, e^{-\lambda t} E^W\left[\left(m(S_\varepsilon^K(t)) + \sum_{k=1}^{N} (h_\varepsilon^K)^{-k} T^k 1(\cdot, t)\right)^2\right] \leqslant C_N (h_\varepsilon^K)^{-2(N+1)}, \tag{5.b}$$

where $S_\varepsilon^K(t) = S_\varepsilon^K(0, t)$. For $p \geqslant 1$, set $\varepsilon_p = \exp(-p^{1/2n})$. Then (5.b) implies

$$\sum_{p=1}^{\infty} \int_0^\infty dt\, e^{-\lambda t} (h_{\varepsilon_p}^K)^{2n} E^W\left[\left(m(S_{\varepsilon_p}^K(t)) + \sum_{k=1}^{N} (h_{\varepsilon_p}^K)^{-k} T^k 1(\cdot, t)\right)^2\right] < \infty.$$

It follows that for $dt$-a.a. $t \in (0; \infty)$,

$$\lim_{p \to \infty} (h_{\varepsilon_p}^K)^n \left(m(S_{\varepsilon_p}^K(t)) + \sum_{k=1}^{N} (h_{\varepsilon_p}^K)^{-k} T^k 1(\cdot, t)\right) = 0, \tag{5.c}$$

where convergence holds $W(d\omega)$ a.s. and in $L^2(\Omega_0, W)$. We may choose $u \in (0; \infty)$ such that (5.c) holds for $t = u$ and for $n = 1, 2, \ldots$. We may also assume that $T^k 1(\cdot, u) \in \mathscr{L}(\Omega_0, W)$ for $k = 1, 2, \ldots$. For this special value $u$, we set $T^k(u)(\omega) = T^k 1(\omega, u)$. Then, for $n = 1, 2, \ldots$,

$$\lim_{p \to \infty} (h_{\varepsilon_p}^K)^n \left(m(S_{\varepsilon_p}^K(u)) + \sum_{k=1}^{n} (h_{\varepsilon_p}^K)^{-k} T^k(u)\right) = 0, \tag{5.d}$$

$W(d\omega)$-almost surely and in $L^2(\Omega_0, W)$. Here we have simply dropped the terms corresponding to $k = n+1, \ldots, N$ in (5.c). Now observe that

$$\lim_{p \to \infty} (h_{\varepsilon_{p+1}}^K)^n - (h_{\varepsilon_p}^K)^n = 0,$$

which implies

$$\lim_{p \to \infty} \sup_{\varepsilon_{p+1} \leqslant \varepsilon \leqslant \varepsilon_p} \left| \sum_{k=1}^{n} (h_\varepsilon^K)^{n-k} T^k(u) - \sum_{k=1}^{n} (h_{\varepsilon_p}^K)^{n-k} T^k(u)\right| = 0, \tag{5.e}$$

almost surely and in $L^2(\Omega_0, W)$. On the other hand, the fact that $m(S_\varepsilon^K(u))$ is a monotone increasing function of $\varepsilon$ (since $K$ is star-shaped) easily implies

$$\begin{aligned}\lim_{p\to\infty} \sup_{\varepsilon_{p+1}\leqslant\varepsilon\leqslant\varepsilon_p} &|(h_\varepsilon^K)^n m(S_\varepsilon^K(u)) - (h_{\varepsilon_p}^K)^n m(S_{\varepsilon_p}^K(u))| \\ &\leqslant \lim_{p\to\infty} ((h_{\varepsilon_{p+1}}^K)^n - (h_{\varepsilon_p}^K)^n)\, m(S_{\varepsilon_p}^K(u)) \\ &\quad + (h_{\varepsilon_p}^K)^n (m(S_{\varepsilon_p}^K(u)) - m(S_{\varepsilon_{p+1}}^K(u))) = 0, \end{aligned} \tag{5.f}$$

again almost surely and in $L^2(\Omega_0, W)$. Combining (5.e) and (5.f) we obtain

$$\lim_{\varepsilon\to 0} (h_\varepsilon^K)^n \left( m(S_\varepsilon^K(u)) + \sum_{k=1}^{n} (h_\varepsilon^K)^{-k}\, T^k(u) \right) = 0, \tag{5.g}$$

a.s. and in $L^2$-norm.

We now define $T^k(t)$ for $t \neq u$. For $r > 0$, $k = 1, 2, ...,$ we set

$$T^k(ru)(\omega) = r \sum_{j=1}^{k} \binom{k-1}{j-1} \left( \frac{\log r}{2\pi} \right)^{k-j} T^j(u)(\omega_r),$$

where $\omega_r$ is defined in Theorem 5.1. Observe that this definition makes sense because the mapping $\omega \to \omega_r$ preserves $W(d\omega)$. Next we verify that (5.g) remains true when $u$ is replaced by $ru$. It is plain that

$$m(S_\varepsilon^K(ru)(\omega)) = r m(S_{r^{-1/2}\varepsilon}^K(u)(\omega_r)). \tag{5.h}$$

On the other hand, noting that

$$h_{r^{-1/2}\varepsilon}^K = h_\varepsilon^K - \frac{\log r}{2\pi} + O(\varepsilon^2),$$

we have

$$\begin{aligned} &\sum_{k=1}^{n} (h_\varepsilon^K)^{-k}\, T^k(ru)(\omega) \\ &= r \sum_{j=1}^{n} \left( \sum_{k=j}^{n} \binom{k-1}{j-1} \left( \frac{\log r}{2\pi} \right)^{k-j} (h_\varepsilon^K)^{-k} \right) T^j(u)(\omega_r) \\ &= r \sum_{j=1}^{n} \left( \left( \frac{h_\varepsilon^K}{h_\varepsilon^K - (\log r)/2\pi} \right) + o((h_\varepsilon^K)^{j-n}) \right) (h_\varepsilon^K)^{-j}\, T^j(u)(\omega_r) \\ &= r \sum_{j=1}^{n} (h_{r^{-1/2}\varepsilon}^K)^{-j}\, T^j(u)(\omega_r) + o((h_\varepsilon^K)^{-n}). \end{aligned} \tag{5.i}$$

Equations (5.h) and (5.i) show that the limiting result (5.g) holds for any $t \in (0, \infty)$, instead of $t = u$, and for $n = 1, 2, \ldots$.

We now verify that the family $((T^k(t))$ satisfies the properties stated in Theorem 5.1. First, using Theorem 4.1, we see that

$$T^k 1(\omega, t) = T^k(t)(\omega), \qquad P(d\omega\, dt) \text{ a.s.},$$

which gives property (i). On the other hand, an easy application of the binomial formula shows that (ii) also holds. Note that, when $t = u$, (ii) is just our definition of $T^k(t)$. The uniqueness of a family satisfying (i) and (ii) is also clear. Indeed, if $(T'^k(t))$ is another family which satisfies both (i) and (ii), we have, for $dt$-a.a. $t$,

$$T'^k(t)(\omega) = T^k(t)(\omega), \qquad W(d\omega) \text{ a.s.} \tag{5.j}$$

We choose $v \in (0, \infty)$ such that (5.j) holds for $t = v$ and for $k = 1, 2, \ldots$. Then the scaling relation (ii) shows that $T^k(t) = T'^k(t)$ for all $k$ and $t$.

At this moment, we have proved Theorem 5.1 and Theorem 5.2 in the case when $K$ is star-shaped. Observe that, although our construction of the family $(T^k(t))$ seems to depend on the choice of $K$, the uniqueness part of Theorem 5.1 shows that it does not.

We now investigate the case when $K$ is not star-shaped. We may choose another compact set $H$, e.g., a disk, such that $H$ is star-shaped and $R(K) = R(H)$. We thus have

$$h_\varepsilon^K = h_\varepsilon^H + O(\varepsilon^2 \log 1/\varepsilon),$$

and it clearly suffices to prove that

$$\lim_{\varepsilon \to 0} (h_\varepsilon^K)^{2n} E[(m(S_\varepsilon^K(t)) - m(S_\varepsilon^H(t)))^2] = 0, \tag{5.k}$$

for every $n \geqslant 1$, $t \geqslant 0$.

For every $N \geqslant 1$, Theorem 4.1 gives the existence of a constant $C_N$ such that, for $\varepsilon \in (0, 1)$,

$$E[(S_\varepsilon^K 1 - S_\varepsilon^H 1)^2] \leqslant C_N (h_\varepsilon^K)^{-2(N+1)}.$$

Let $N = 8n$ and $\varepsilon_p = \exp(-p^{1/4n})$. Arguing as in the beginning of the proof, we can find $u \in (0, \infty)$ such that

$$\lim_{p \to \infty} (h_{\varepsilon_p}^K)^{2n} E[(m(S_{\varepsilon_p}^K(u)) - m(S_{\varepsilon_p}^H(u)))^2] = 0. \tag{5.l}$$

Now, by scaling,

$$(m(S_{\varepsilon_p}^K(u)), m(S_{\varepsilon_p}^H(u))) \overset{(d)}{=} \varepsilon_p^2 (m(S_1^K(t_p)), m(S_1^H(t_p))),$$

where $t_p = \varepsilon_p^{-2} u$. Thus (5.1) implies

$$\lim_{p \to \infty} \frac{(\log t_p)^{2n}}{t_p^2} E[(m(S_1^K(t_p)) - m(S_1^H(t_p))^2] = 0. \tag{5.m}$$

The point is that $m(S_1^K(t))$ is a monotone increasing function of $t$. Moreover,

$$\begin{aligned}
&\frac{(\log t_p)^{2n}}{t_p^2} E[(m(S_1^K(t_{p+1})) - m(S_1^K(t_p)))^2] \\
&\leqslant \frac{(\log t_p)^{2n}}{t_p^2} E[m(S_1^K(t_{p+1} - t_p))^2] \\
&\leqslant C \frac{(\log t_p)^{2n}}{t_p^2} \frac{(t_{p+1} - t_p)^2}{(\log(t_{p+1} - t_p))^2} \xrightarrow[p \to \infty]{} 0.
\end{aligned}$$

Therefore (5.m) implies

$$\lim_{t \to \infty} \frac{(\log t)^{2n}}{t^2} E[(m(S_1^K(t)) - m(S_1^H(t)))^2] = 0,$$

and (5.k) follows, again by scaling. This completes the proof of Theorem 5.2. ∎

(5.3) Spitzer [21] obtains an asymptotic expansion for the quantity

$$E_K(t) = \int dy\, P_y[T_K \leqslant t].$$

It is clear that

$$\begin{aligned}
E_K(t) &= \int dy\, P[T_K(y) \leqslant t] = E[m(S_1^K(0, t))] \\
&= tE[m(S_{t^{-1/2}}^K(0, 1))].
\end{aligned} \tag{5.n}$$

Setting $\varepsilon = t^{-1/2}$ and taking expected values in Theorem 5.2 lead to a new proof and an improvement of Spitzer's result. We fix $\lambda = 1$. Then,

$$E[m(S_\varepsilon^K(0, 1))] = -\sum_{k=1}^{n} (h_\varepsilon^K)^{-k} E[T_k(1)] + O((h_\varepsilon^K)^{-n-1}), \tag{5.o}$$

where

$$h_\varepsilon^K = -\frac{1}{\pi} \log \frac{1}{\varepsilon} - \frac{1}{\pi}\left(\frac{1}{2} \log 2 - \gamma - R(K)\right) + O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right).$$

The quantities $E[T_k(1)]$ can be computed as follows. By Theorems 3.1 and 5.1, for $k \geqslant 2$,

$$\begin{aligned} E[T_k(\zeta)] = 0 &= \int_0^\infty dt\, e^{-t} E[T_k(t)] \\ &= \int_0^\infty dt\, e^{-t} t \sum_{l=1}^{k} \binom{k-1}{l-1} \left(\frac{\log t}{2\pi}\right)^{k-l} E[T_l(1)] \\ &= \sum_{l=1}^{k} \alpha_l^k E[T_l(1)], \end{aligned}$$

where

$$\alpha_l^k = (2\pi)^{-(k-l)} \binom{k-1}{l-1} \int_0^\infty dt\, t(\log t)^{k-l} e^{-t}.$$

An integration by parts shows that, for $p \geqslant 1$,

$$\begin{aligned} \int_0^\infty dt\, t(\log t)^p e^{-t} &= \int_0^\infty dt(\log t)^p e^{-t} + p \int_0^\infty dt(\log t)^{p-1} e^{-t} \\ &= \Gamma^{(p)}(1) + p\Gamma^{(p-1)}(1). \end{aligned}$$

The quantities $\Gamma^{(p)}(1)$ can be computed in terms of $\gamma$ and the numbers $\zeta(j)$, $j = 2, 3, \ldots$ (where $\zeta(\cdot)$ denotes the Riemann function, see [17, pp. 40–44]). Let us restrict our attention to the first three terms of the expansion of $E_k(t)$. We only need to know

$$\Gamma'(1) = -\gamma, \qquad \Gamma''(1) = \gamma^2 + \zeta(2) = \gamma^2 + \pi^2/6,$$

from which it follows that

$$E[T_2(1)] = \frac{1}{2\pi}(\gamma - 1), \qquad E[T_3(1)] = \frac{1}{4\pi^2}\left(\gamma^2 - 2\gamma + 2 - \frac{\pi^2}{6}\right).$$

Hence, coming back to (5.o),

$$\begin{aligned} E[m(S_\varepsilon^K(0, 1))] = &-(h_\varepsilon^K)^{-1} - \frac{1}{2\pi}(\gamma - 1)(h_\varepsilon^K)^{-2} \\ &- \frac{1}{4\pi^2}\left(\gamma^2 - 2\gamma + 2 - \frac{\pi^2}{6}\right)(h_\varepsilon^K)^{-3} + O((h_\varepsilon^K)^{-4}), \end{aligned}$$

and using (5.n) we obtain the first three terms of the expansion of $E_K(t)$. The first two terms already give Theorem 2 of Spitzer [21] (Spitzer's notation is slightly different from ours: what we call $R(K)$ is $-2R(K)$ for Spitzer).

It is worth noting that the expansion for $E_K(t)$ depends only on the quantity $R(K)$. This should be compared with similar expansions in higher dimensions [15], which involve integrals with respect to the equilibrium measure of $K$.

## Acknowledgment

I thank Professor E. B. Dynkin for several useful conversations.

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